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No. 778.

THE STRENGTH OF PILLARS.—AN ANALYSIS.

By LEOPOLD EIDLITZ, Esq.

PRESENTED MARCH 4TH, 1896.

WITH DISCUSSION.

When a pillar is compressed endwise by a force gradually increased to the breaking point, it is reasonable to assume that at the intrados of the pillar at the point of greatest deflection a strain has been set up which is as great as the stress causing rupture in a specimen of the material, the height or vertical length of which specimen does not materially exceed its smallest diameter. Without entering upon the question of the magnitude of this excess, which has been variously estimated, it may be safely placed at that point where the strain generated by the element of bending becomes inappreciable, and it is probable that endwise compression is always accompanied with bending strains. The questions examined, however, in this paper are mainly, what is the maximum strain in pillars compressed endwise in the center of resistance by weights less than the breaking weights, or by weights not applied in the center of resistance.

Experiment shows that where a given pillar is deflected under increasing loads, the deflections increase at a greater ratio than the loads causing them, and the reverse is also true. The less the compressing force, the greater the ratio of reduction in the deflection.

The maximum strain at the middle of the intrados of a pillar bent to the breaking point may be assumed to be C, the ultimate resistance of the material to crushing. Let W be the breaking load of a solid 1-in. square pillar; the moment of resistance to bending is as follows :

If the vertical strain of the breaking load is deducted from the maximum strain C, the quantity $C - W$ remains to resist the bending moment $\delta\ W$, where δ is the breaking deflection and $\frac{d^3}{6}(C - W) = \delta\ W$, and as $d = 1$ the total strain $C = W(1 + 6\ \delta)$.

Now, let S be a strain $S < C$ and $W_1 < W$ and $\delta_1 < \delta$, then $S = W_1 \times (1 + 6\ \delta_1)$. Compare these two equations, and it appears that the relation of S to C can be as the respective weights W_1 to W only on the assumption that $1 + 6\ \delta_1 = 1 + 6\delta$, or that $\delta_1 = \delta$. It seems clear, therefore, that safe loads of pillars are not proportional to the breaking weights, but must be referred to a permissible maximum strain.

In building, the practice still prevails of considering cast-iron pillars planed top and bottom as subject to compound flexure as perceived in the testing machine. That this is not the case needs no special argument here, as the fact is universally recognized.

Breaking loads are computed by Gordon's and other formulas based upon Hodgkinson's experiments. In these experiments endwise compression was applied to the specimen as nearly as practicable in the center of resistance, or at least in the center of gravity of the pillar tested, which is also the center of resistance, unless the material is heterogeneous.

On pages 382 and 383 of this paper is an analysis of the bending strains caused by eccentric loads. The exact relation of the eccentric breaking load to the concentric breaking load is expressed in a formula.

Endwise Compression.

A specimen of cast-iron 1 in. square is extended 0.0017 of its length by a force of about 16 000 lbs. applied at the ends, and when so extended it is torn asunder. If the specimen is 5 ins. long, its extension

under a tensile strain of 16 000 lbs. is $5 \times 0.0017 = 0.0085$; when 10 ins. long, it extends under the same strain $10 \times 0.0017 = 0.017$; and when 100 ins. long, it is extended $100 \times 0.0017 = 0.17$ in., but in each event it requires a strain of neither more nor less than 16 000 lbs. to cause rupture.

A short specimen of cast-iron of the same sectional area of 1 in. is crushed by a compressing force of about 96 000 lbs.; but as the specimen becomes longer, the force required to disintegrate it becomes rapidly less. Thus, when the specimen is 5 ins. long, it will break under 85 000 lbs.; when 10 ins. long, under 45 000 lbs.; when 15 ins. long, under 30 000 lbs.; when 20 ins. long, under 23 000 lbs.; when 40 ins. long, under something less than 8 000 lbs.; and when 80 ins. long, under 2 000 lbs.

Now the main difference in the behavior of two specimens acted upon by compression and tension is that the latter specimen remains in a straight line during the application of force, while the former becomes curved. It is said, therefore, that a pillar or strut bends or deflects under endwise compression. Unequal compression, though not a primary cause, is a result of the process of bending. This involves a brief consideration of the nature of elasticity.

Elasticity is the internal force which in matter persists in maintaining atomic or molecular relations. The sum of persistence which a given kind of matter is capable of constitutes its potential elasticity. External force applied to given matter brings into activity an equal amount of elastic force, and while this external force remains in action, the potential energy of elasticity of the matter in question is reduced by that amount. When the external force is equal to the internal potential resistance of elasticity, the specimen is on the point of disruption. Any additional amount of external force, no matter how small, will cause disruption.

The exertion of the elastic force inherent in matter is always accompanied with fatigue. This means that the potential energy of elastic force is reduced by exertion. The amount of fatigue caused by comparatively small increments of external force (as compared in magnitude with the total potential energy of the elastic force of a given material) is imperceptible to the senses unless scrutinized by special sensitive instruments of measurement. The extent to which this is the case is known as the elastic limit of the material, and the amount

of fatigue which becomes perceptible beyond the elastic limit is called the permanent set. Within the elastic limit the increments of compression and extension are nearly the same for every equal amount of additional external force applied, so much for every additional 1 000 lbs. This is called uniform elasticity. Beyond the elastic limit, compression and extension increase with every additional equal amount of external force applied. This is called variable elasticity.

Within the elastic limit each pound of force exerted results in an equal amount of compression, or an equal amount of extension, though the amounts of compression and extension are never the same in any one material. In matter uniformly elastic, the number of pounds of external force divided by the resulting compression is a constant, or, if divided by the resulting extension, the quotient will be another constant. These constants are called the moduli of elasticity of the given material. So each material has its special moduli of elasticity for compression and for extension, and if the modulus of elasticity is expressed by E, the weight by W, the amount of compression by C, and that of extension by T, the modulus of elasticity for compression by E_c and that for tension by E_t, it follows:

$$\frac{W}{C} = E_c \quad \text{and} \quad \frac{W}{T} = \mathrm{E}_t$$

$$W = E_c\, C \quad \text{“} \quad W = E_t\, T$$

$$\frac{W}{E_c} = C \quad \text{“} \quad \frac{W}{E_t} = T$$

Within the range of variable elasticity each number of pounds of strain has its special modulus of elasticity; hence, the convenience of its use in analysis is lost, and stress must be referred directly to the amount of the compression or extension caused by it in matter.

The compression per square inch area caused in cast-iron by weights ranging from 1 000 to 96 000 lbs., and those of wrought iron from 1 000 to 52 000 lbs., as presented in Tables Nos. 1 and 2, are derived from curves based upon the results of Hodgkinson's experiments.

Table No. 1 gives in column 2 the compression of 1 in. area of cast-iron for weights from 96 000 to 1 000 lbs., also in column 3 the compression due to the last 1 000 lbs. applied. It will be seen from the last that the elasticity of compression of cast-iron is variable throughout.

Table No. 2 gives in column 2 the compression of 1 in. area of wrought iron for loads given in column 1; also in column 3 the compression due to the last 1 000 lbs.

It will be seen that for wrought-iron pillars the additional increment of compression for every additional 1 000 lbs. is 0.00004 up to 12 000 lbs., and 0.00005 from 12 000 to 24 000 lbs. Twenty-four thousand to 28 000 lbs., therefore, is accepted as the limit of elasticity of wrought iron. The average increase per 1 000 lbs. up to 24 000 lbs. is 0.000045; therefore, $1\,000 \div 0.000045$ gives the modulus of elasticity for the compression of wrought iron as 22 000 000.

For the extension of wrought iron within the elastic limit the modulus is found by experiment to vary between 25 000 000 and 28 000 000.

Deflection.

For greater convenience of analysis and of arriving at tables which give the breaking and safe weights per inch area for lengths expressed in terms of diameters or radii of gyration, let the diameter of the pillars be $d = 1$; let them be solid and square in form; hence, the sectional area will also be $a = 1$; the moment of inertia will be $\frac{1}{12}$; the moment of resistance $\frac{1}{6}$, and the pillar length expressed in diameters will represent so many inches.

If a pillar is compressed vertically by its breaking weight and is in some way prevented from bending, its potential elastic energy for compression still remaining is $C - W$, where C is the crushing weight of the material, and the potential energy of tension still remaining is $T + W$ when T represents the ultimate resistance of the material to tension.*

If, now, the pillar is left free to bend, then at the breaking point this remaining potential energy is exhausted; hence, the bending moment $\delta\ W$, δ being the deflection, is in equilibrium with the potential energy multiplied by the moment of resistance; hence,

$$\delta\ W = \frac{1}{6}(C - W) = \frac{1}{6}(T + W) \text{ and } \delta = \frac{C - W}{6\ W} = \frac{T + W}{6\ W}$$

* When a body is compressed or extended by a force less than its ultimate resistance to compression or tension, the balance of compression or extension possible may be considered to be its potential energy for resisting another external force.

Let a body capable of an ultimate resistance to compression C, or of resistance to extension T, be compressed in the one instance by a force W, or extended by a force W, in either event its potential energy will be $C - W$ or $T - W$. But if the external force is one of compression, in either case the potential energy of resistance to compression is $C - W$, and that to extension is $C + W$.

Under conditions of uniform elasticity this value of δ is absolutely correct. But inasmuch as elasticity is variable, it would be true only if the horizontal sectional planes drawn transversely to the pillar length were to become curved during the process of bending.

To ascertain exact deflections related to certain breaking loads from an assumed condition of compression, or of compression and tension at the middle of a pillar bent to the breaking point, the following process is practicable:

Let $a\ b\ c\ d$ (Fig. 1) represent a diagram of compression at the middle of a square cast-iron pillar 1 in. on a side bent to the breaking point. The stress at b, the intrados of the pillar, is assumed to be 96 000 lbs., the crushing stress of the material. The compression due to a stress of 96 000 lbs., as given in the table, is 0.026. At the point a, the extrados of the bent pillar, the compressive strain is assumed to be 32 000 lbs. Compression at the intervening points between a and b is as marked upon the vertical lines drawn at those points, and the strain due to these respective compressions, as given in Table No. 1, is the number of pounds marked at the bottom of the vertical lines. $A\ B\ C\ D$ (Fig. 1) represents a strain diagram at the middle of the pillar divided into ten laminas representing the mean stress of each lamina. This diagram gives by computation an area of 71 080 lbs. and a center of gravity 0.0692 distant from the axis of the pillar, which distance constitutes the deflection.

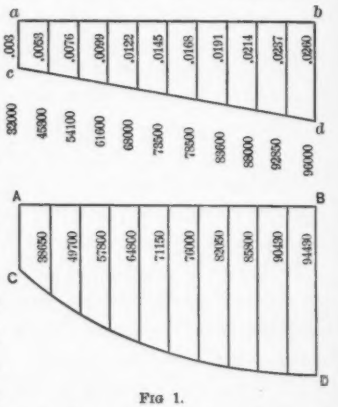

FIG 1.

A series of diagrams like these with varying strains at a forms a basis for tables giving the breaking weight, deflection, and, as will be shown hereafter, the breaking radius and breaking length of all possible pillars.* Tables Nos. 3 and 4 have been prepared in this

* This process of ascertaining the breaking weights, deflections, radii and corresponding breaking lengths, as given in Tables Nos. 3 and 4, is absolutely correct. The method first stated (page 123) of computing deflection by the formula $\delta = \frac{C - W}{6\ W}$ is less accurate and so is the resulting determination of the radius, and consequently of the pillar length. The result is that for given pillar lengths the breaking weights are less than those based upon

manner; Table No. 3 is for cast-iron pillars, and Table No. 4 for wrought-iron pillars.

Radius.

Let $a\,b\,c\,d\,g$ (Fig. 2) represent the strain diagram at the middle of a pillar bent to the breaking point under a breaking load W $a\,b = T$ (tension) and $d\,g = C$ (the ultimate crushing resistance of the material). Then $m\,b\,o\,n\,g$ represents the total resistance to bending, and $a\,d\,n\,m$ the resistance to vertical compression.

If the latter is assumed to pervade the whole pillar, the former determines the varying positive and negative strains and their accompanying compression and extension, which constitute the elements that determine the radius.

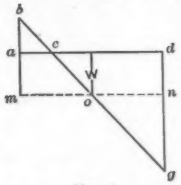

Fig. 2.

At the intrados, the pillar always shortens in bending. At the extrados, it may shorten less, not at all, or it may lengthen by tension. The shortening at the breaking point at the middle of the pillar is $d\,g = c$. That part of $d\,g$ which represents resistance to bending ($n\,g$) becomes continually less at points approaching the top and bottom of pillar, and is nil at the top and bottom.

The average of the strains in point (if the bending curve of the pillar is assumed to be the same as that of a rectangular beam under a transverse load in its center) is $p\,(C - W) = 0.6125 \times (C - W)$. A circular curve drawn through the top, the bottom, and the middle of the intrados of a pillar at the breaking point therefore represents a uniform strain of compression of $W + p\,(C - W)$, and a similar circular curve at the extrados represents a uniform strain of

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the assumption of a curved resistance diagram at the middle of the pillar at the point of breaking. Exactly how much this difference amounts to may be ascertained by comparing Tables Nos. 1 and 2 with Tables Nos. 3 and 4.

Figs. 3 and 4 show the curves of breaking weights for respective pillar lengths in accordance with Tables Nos. 1 and 3 and Nos. 2 and 4 for more convenient comparison. But inasmuch as breaking weights are of themselves no criterion of safe loads, the formula $\delta = \frac{C - W}{6\,W}$ has been used in the following analysis of deflections under loads less than the breaking weight with results that show no material difference when applied to either of Tables Nos. 1 and 3 or Nos. 2 and 4, except in the case of long wrought-iron pillars, which show somewhat increased safe loads when computed in accordance with Table No. 4.

Inasmuch as the breaking loads for both cast and wrought iron, as given in Tables Nos. 1 and 2, more nearly coincide with the results of Hodgkinson's experiments, and inasmuch as a moderate reduction of the safe loads of long columns secures reasonably moderate deflections, it has been deemed best to compute the tables of safe loads in accordance with Tables Nos. 1 and 2.

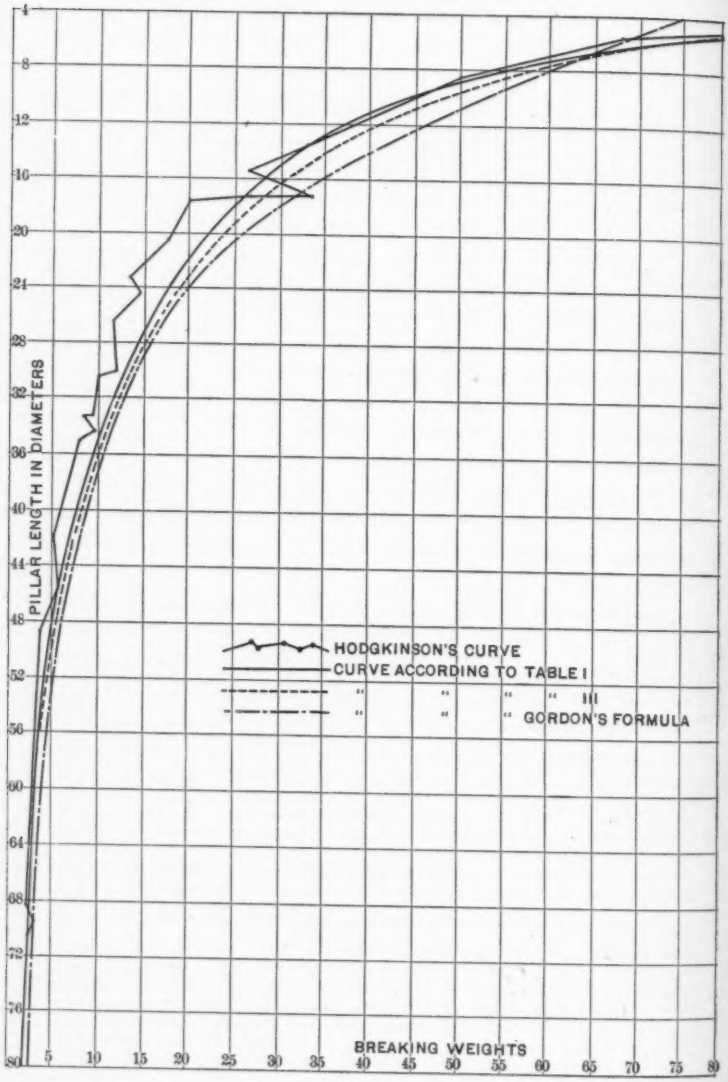
PILLAR LENGTH IN DIAMETERS
4
8
12
16
20
24
28
32
36
40
44
48
52
56
60
64
68
72
76
80
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
BREAKING WEIGHTS
HODGKINSON'S CURVE
CURVE ACCORDING TO TABLE I
" " " " III
" " " GORDON'S FORMULA

FIG. 3.

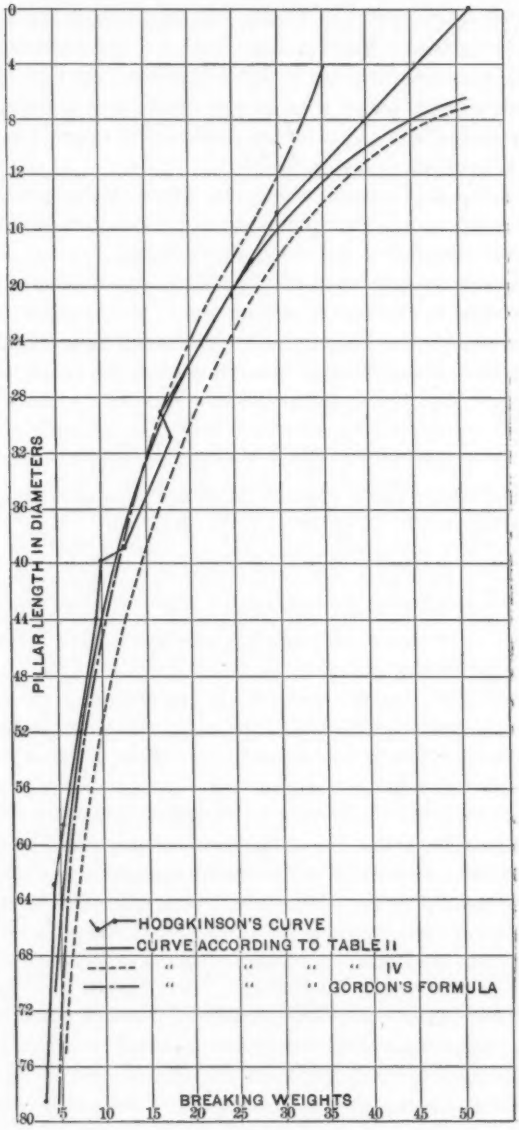
PILLAR LENGTH IN DIAMETERS
0
4
8
12
16
20
24
28
32
36
40
44
48
52
56
60
64
68
72
76
80
BREAKING WEIGHTS
5
10
15
20
25
30
35
40
45
50
HODGKINSON'S CURVE
CURVE ACCORDING TO TABLE II
" " " IV
" " " GORDON'S FORMULA

FIG. 4.

$W - p\,(C - W)$ which may be negative or positive, a strain of compression or tension. The radius and deflection of a pillar are the sole elements of its length (as will be shown hereafter), and though the circular curve which passes through the middle and top and bottom differs from the bending curve, this difference of length may be neglected without appreciable error.

The radius of a circular curve is the radius of any portion of it, being dependent upon the compression at the intrados, and the compression or extension at the extrados. If compression be expressed by the sign C_p and extension by the sign E_x and $a\ b\ c\ d$,

FIG. 5.

Fig. 5 represents a portion of the pillar in a circular curve, of which $f\ g$ is the axis, then $o\ g$ is the radius R, and if the diameter of the pillar is d, then $a\ c;\ b\ d = R + \frac{d}{2} = R - \frac{d}{2}$,

$$\text{and if } d = 1 \text{ then } R = \frac{a\ c + b\ d}{2\ (a\ c - b\ d)}.$$

Let the height of $a\ b\ c\ d$ be considered as 1 prior to deformation by bending, then $a\ c = 1 - C_p\,[W - p\,(C - W)]$, and

$$b\ d = 1 - C_p\,[W + p\,(C - W)] \text{ and}$$

$$R = \frac{2 - [C_p\,[W - p\,(C - W)] + C_p\,[W + p\,(C - W)]]}{2\,[C_p\,[W + p\,(C - W)] - C_p\,[W - p\,(C - W)]]}$$

When $p\,(C - W)$ is greater than W, then $W - p\,(C - W)$ becomes negative and denotes tension.

Example.—To find the radius for a breaking weight $W = 48\,000$ lbs., the ultimate resistance of the material to crushing being $C = 96\,000$ lbs. $C - W = 48\,000$ and $p\,(C - W) = 29\,400$; $W + p\,(C - W) = 48\,000 + 29\,400 = 77\,400$ lbs., $W - p\,(C - W) = 18\,600$ lbs. By Table No. 1 the compression for 77 000 lbs. is 0.016045. For the additional 400 lbs. at the rate of the last 1 000 lbs., which is 0.000454; it is 0.000182; therefore, for 77 400 lbs. the compression is $0.016045 + 0.000182 = 0.016227$. The compression due to 18 600 lbs., the uniform stress at the extrados by Table No. 1, is 0.001606; hence,

$$R = \frac{2 - (0.001606 + 0.016227)}{2\,(0.016227 - 0.001606)} = \frac{2 - 0.017833}{2\,(0.014621)} = 67.7.$$

Table No. 1 gives the uniform strains for cast-iron at the intrados in column 6 and those of the extrados in column 7. Column 4 of the same table gives the strain at the middle of the extrados. It will be observed that this strain is one of compression throughout the pillar

area when the breaking weights range from 96 000 lbs. down to 48 000 $= \frac{1}{2} C$; at 48 000 lbs. the stress in the middle of the pillar at the extrados is nil. From 48 000 down to 40 000 lbs. breaking weight, the strains at the middle at the extrados are tensional, beginning with 0 and ranging up to 16 000 lbs., the assumed ultimate resistance of the material to tension. At the intrados in the middle of the pillar down to a breaking load of 40 000 lbs. the strain is constant at the ultimate resistance of the material to crushing, which is assumed at 96 000 lbs. When cast-iron pillars exceed a length of 11.3 diameters, they break under a load less than 40 000 lbs. by rupture at the extrados; hence, the strain at the middle of the intrados becomes less than 96 000 lbs., and, as is shown by Table No. 1, when a 1-in. square cast-iron pillar is 115.1 ins. long, it breaks by tension at the extrados, while at the middle of the intrados the strain of compression has run down to 18 000 lbs.

It may be noticed also that deflection becomes greater when the pillar becomes longer and the breaking weight becomes less. As the breaking weights descend from 96 000 to 1 000 lbs., deflection begins with 0.00175 and ends with 2.83333. The radius, on the other hand, is largest with the greatest breaking weight and becomes less when the breaking weight becomes less. It is least at a breaking weight of 48 000 lbs. From this point downward in the scale of breaking weights the radius becomes larger again.

Both radius and deflection depend upon the resistance of the pillar to bending at the breaking point. This resistance is the potential energy of the material still remaining after the vertical force of endwise compression is deducted from the ultimate strength of the material; $C - W$ or $T + W$ are the measure of this potential energy. The greater the elastic force of $C - W$ or $T + W$, the greater the deflection.

This accounts for the fact that pillars break with smaller deflections than beams under a transverse load. In the latter, resistance to bending is the total ultimate strength of the material, while in the former a portion of this strength is consumed by endwise vertical compression.

Length of a Pillar.

Let $a\,b$ (Fig. 6) represent a small portion of a circular curve, the radius of which is $a\,o$, and its diameter $a\,d$; then the curve $a\,b$ may without serious error be considered as a straight line, and the triangle

$a\,c\,b$ is homologous to the triangle $a\,b\,d$ and $a\,c : a\,b = a\,b\,;\, b\,d$ and $a\,c \times b\,d = a\,b^2$.

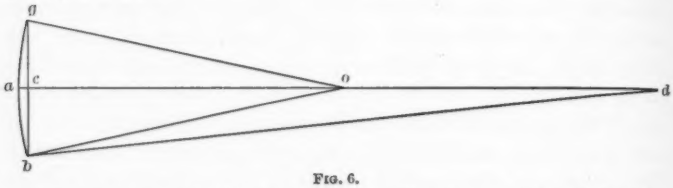

Fig. 6.

Let $b\,g$ represent the length of a pillar bent with a deflection $\delta = a\,c$ and $b\,g = L$, hence $L = 2\,a\,b$ and $a\,b^2 = \frac{L^2}{4}$ and $b\,d = 2\,R$; then by substituting these values of $a\,c$, $b\,d$ and $a\,b$ in $a\,c \times b\,d = a\,b^2$.

$$2\,R\,\delta = \frac{L^2}{4} \text{ and } L^2 = 8\,R\,\delta \text{ and } L = \sqrt{8\,R\,\delta}.$$

Tables Nos. 1, 2, 3 and 4 give the length of pillars corresponding to the respective breaking weights for solid square cast and wrought-iron pillars, on the assumption of straight and curved diagrams of resistance at the middle of the pillar. Fig. 3 gives resistance curves at the breaking point of solid square cast-iron pillars of all lengths and breaking weights in accordance with Tables Nos. 1 and 3, also with Gordon's formula and Hodgkinson's experimental results. Fig. 4 gives the same for solid square wrought-iron pillars.

Eccentric Loads.

The foregoing analysis refers to pillars compressed endwise by a force acting in the center of gravity of the resistance, which in a homogeneous pillar is also the center of gravity of its cross-section. In that case stress is assumed to be equally distributed at the top of the pillar, also at the bottom.

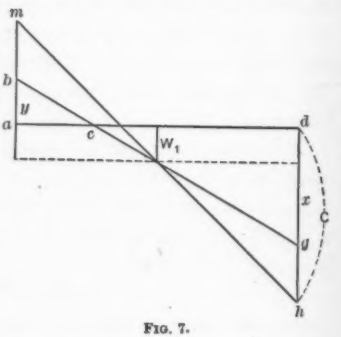

Fig. 7.

When the force compressing the pillar is not acting in its center, and the eccentricity is n (a fraction of its diameter), then the distribution of stress at the top of the pillar is not uniform, but may be expressed by the diagram, Fig. 7, where W_1 being 1, x denotes strain

of compression at the intrados x, 1 or $x\ W$, and y the strain at the extrados, y, 1 or $y\ W_1$; the latter may be a strain of compression or tension or nil, and

$$\frac{x-1}{6} = \frac{y+1}{6} = n, \text{ and}$$

$$x = 6\,n + 1 \text{ and } y = 6\,n - 1.$$

At the point of breaking, the strain at the intrados in the middle of the pillar has increased to C, the crushing strain of the material, hence the moment due to deflection δ_1 is resisted by $\frac{g\ h}{6}$. The resistance to the total bending is therefore

$$\frac{C - W_1}{6} = (n + \delta_1)\ W_1$$

Let $a\ b\ c\ g$ (Fig. 8) represent the upper half of the pillar bent by the load W_1, placed upon the top of the pillar at the distance n from its center of resistance (where n represents a fraction of the diameter of the pillar), and let δ be the resultant deflection and $\frac{C - W_1}{6}$, the resistance to bending

$$\frac{C - W_1}{6} = (n + \delta_1)\ W_1 \ldots\ldots (1)$$

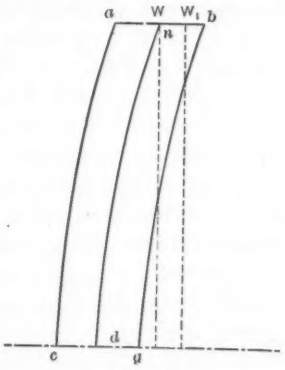

FIG. 8.

Let W, the breaking weight of the pillar in question, when acting in the center of resistance, be placed at the center, the moment of resistance to bending will be $\frac{C - W}{6}$ and the pillar will bend with a deflection $\delta < \delta_1$ and $\delta\ W : \delta_1\ W = C - W : C - W_1$ or $\delta : \delta_1 = C - W : C - W_1$ and $\delta_1 = \delta\ \frac{C - W_1}{C - W}$, but as $\delta = \frac{C - W}{6\ W}$, then $\delta_1 = \frac{C - W_1}{6\ W}$.

When this value of δ_1 is introduced in equation (1), then

$$W_1 = \pm \sqrt{\left(\frac{C + W + 6\,n\ W}{2}\right)^2 - C\ W} + \frac{C + W + 6\,n\ W}{2}$$

In the case of the cast-iron pillars, the bending resistance is $\frac{T + W}{6}$, then

$$W_1 = \pm \sqrt{\left(\frac{W - 6\,n\ W - T}{2}\right)^2 + T\ W} + \frac{W - 6\,n\ W - T}{2}.$$

Table No. 5 gives the breaking weights for cast-iron pillars of length, in diameters varying from 5 to 50, when compressed by loads placed to one side of their neutral axis, for eccentricities of 0.1, 0.3, 0.5, 0.7 and 1.0. Table No. 6 gives breaking loads for wrought-iron pillars for lengths in diameters from 7 to 44 for the same eccentric loads.

THE DEFLECTION AND RADIUS CAUSED BY $W_1 < W$ WHEN PLACED IN THE CENTER OF RESISTANCE.

Let a pillar of the length L, breaking deflection δ, breaking weight W, and breaking radius R, be bent to a deflection $\delta_1 = \frac{\delta}{2}$ or $\delta_{11} = \frac{\delta}{4}$; the respective weights will be less than W, say $W - a$, and $W - (a + b)$ and the corresponding bending moments are $\delta_1 (W - a)$ and $\delta_{11} (W - a - b)$ and the bending radii R_1 and R_{11}. Inasmuch, however, as L is constant, then $L^2 = 8\,\delta\,R = 8\,\delta_1\,R_1 = 8\,\delta_{11}\,R_{11}$ or $\delta\,R = \delta_1\,R_1 = \delta_{11}\,R_{11} = \ldots \delta_n\,R_n$. The value of $R\,R_1\,R_{11}$ is always a function of the respective deflections, δ, δ_1 and δ_{11}, irrespective of bending weight.

The bending moment, on the other hand, depends on the bending weight as well as on the deflection. The greatest bending strain occurs at the middle of the intrados of the pillar, and at the breaking point it is C, the crushing resistance of the material when deflected to the point δ. With a deflection δ_1, the greatest stress at the middle of the intrados will be $S < C$, and with a deflection δ_{11} it is $S_{11} < S$.

Comparing the bending moments with the respective resistance, $\frac{C - W}{6} = \delta\,W$; $\frac{S - W_1}{6} = \delta_1\,W_1 = \delta_1 (W - a)$ and

$$\frac{S_{11} - (W - a - b)}{6} = \delta_{11} (W - a - b) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If elasticity is assumed to be uniform, and $S - W$ is the internal stress which resists bending and the compression due to $S - W$ is $\frac{S - W}{E}$, E being the modulus of elasticity of the material in question, the radius may be found by comparing the length of any unit of measure at the intrados with a corresponding unit of the pillar axis, without regard to the vertical compression which is uniformly distributed. In that case compression at the pillar axis is nil and at the intrados it is $\frac{S - W}{E}$ as stated previously, and $1 - \frac{(S - W_1)}{E} : 1 = R_1 - \frac{d}{2} : R_1$ and the radius $R_1 = \frac{1E}{2\,p\,(S - W_1)}$, hence

$$R_1 = \frac{E}{2p\,(S-(W-a))} \text{ and } R_{11} = \frac{E}{2p\,(S_{11}-(W-a-6))} \text{ and}$$

$$R_1 : R_{11} = S_{11} - (W-a-b) : S_1 - (W-a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

From equation (2):

$$S = (W-a)\,(1+6\,\delta_1) \text{ and } S_{11} = (W-a-b)\,(1+6\,\delta_{11})$$

and as $\delta_1 = \frac{C-W}{12\,W}$ and $\delta_{11} = \frac{C-W}{24\,W}$

$$S = (W-a)\left(1+\frac{C-W}{2\,W}\right) \text{ and}$$

$$S_{11} = (W-a-b)\left(1+\frac{C-W}{4\,W}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Substitute values of S and S_{11} in equation 3.

$$R_1 : R_{11} = (W-a-b)\left(1+\frac{(C-W)}{4\,W}\right) - (W-a-b):$$

$$(W-a)\left(1+\frac{C-W}{2\,W}\right) - (W-a)$$

$$R_1 : R_{11} = (W-a-b)\,\frac{(C-W)}{4\,W} : (W-a)\,\frac{(C-W)}{2\,W}$$

$$R_1 : R_{11} = \frac{(W-a-b)}{2} : W-a$$

$$R_{11} = 2\,R_1\,\frac{W-a}{(W-a-b)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

But inasmuch as $R_{11}\,\delta_{11} = R_1\,\delta_1$ and $\delta_1 = 2\,\delta_{11}$, hence $2\,R_1$ must be $= R_{11}$ in all cases. Therefore equation (5) is true only when a and b are nil.

At all points short of the breaking deflection it is known by experiment that there is no equilibrium during the process of bending under a breaking weight until the breaking deflection is reached, and that resistance to the bending moment is least at the smallest deflection, and becomes greater as the breaking deflection is approached.

Yet at half the breaking deflection $\delta_1 = \frac{\delta}{2}$

$$\delta_1\,W = \frac{S-W}{6};$$

at the breaking point $\delta\ W = \frac{C-W}{6}$; hence,

$$S-W : C-W = \delta_1\,W : \delta\,W = 1:2, \text{ and}$$

$$S-W = \frac{C-W}{2}.$$

If $S - W$ represented an internal force of resistance to the bending moment, then experiment would show a condition of equilibrium ; but this not being the case $\frac{S - W}{6}$ only partially consists of compression, a compression due to a strain less in magnitude than $S - W$, in fact a stress equal to $(W - a)$.

But $S - W$ is the true measure of the radius, hence the radius is the result of two forces. The one is $\delta_1 (W - a)$ which is the force of compression at the intrados in excess of that at the pillar axis, and the force $\delta_1 a$, which for the present may be attributed to the vertical stress of a weight $W - a$ without a definition of its exact nature, simply because endwise compression is the sole cause of any and all deformation of the pillar. An example will illustrate the foregoing more clearly:

Given a solid 1-in. square cast-iron pillar and 9.5 ins. long, then $C = 96\,000$ lbs., $W = 48\,000$ lbs., $\delta = 0.16666$, $R = 67.7$ and $\delta R = 11.3$; consequently any $\delta_1 R_1$ must also be equal to 11.3, and if $\delta_1 = \frac{\delta}{2} = 0.08333$, then $R_1 = 135.4$.

The value of W_1 being unknown further than that it is less than W, it must be something between 48 000 lbs. and nil. Whatever its magnitude; when $\delta_1 = \frac{\delta}{2}$, the radius R_1 must be 135.4.

Let W be 48 000 lbs., then $\delta_1 W = \frac{\delta W}{2}$ and as $\delta W = \frac{C - W}{6}$ then $\frac{\delta_1}{2} W = \frac{C - W}{12} = \frac{S - W}{6}$ where S is the total strain at the intrados when the deflection is $\delta_1 = 0.08333$, and $\delta_1 W = \frac{S - W}{6} = 4\,000$ lbs., and $S - W = 24\,000$ lbs.; hence $S = 72\,000$ lbs. S being known, the radius may be computed by the formula developed on page 380, by substituting S for C and

$$R_1 = \frac{2 - [C_p (W - p (S - W)) + C_p (W + p (S - W))]}{2 [C_p (W + p (S - W)) - C_p (W - p (S - W))]} = 140.3.$$

If $W = 44\,000$ lbs., $S = 66\,000$ lbs. and $R_1 = 172.7$.

If $W = 40\,000$ lbs., then $S = 60\,000$ lbs. and $R_1 = 219.5$.

If $W = 39\,070$ lbs. (which is, as will appear hereafter, to be the true weight which bends the pillar with a deflection $\delta = 0.08333$), then $S - W = 19\,534$ and $S = 58\,606$ and $R_1 = 233.1$.

Table No. 7a gives the values of S, $(S - W)$ and R for possible W_1 from 48 000 down to 24 000 lbs.

Inasmuch as the radius corresponding with a deflection $\delta_1 = 0.08333$

cannot be more than 135.4, it appears that whatever the bending weight which results in a resistance of $\frac{S - W_1}{6}$ the radius to correspond with that resistance is less than the radii shown in Table No. 7A, whence $S - W_1$ must be correspondingly greater than those appearing in the table.

Whatever the value of $S - W_1$, or X as it will be called hereafter, it is greater than the bending moment.

It constitutes the resistance to a force acting transversely to the pillar length to the extent of the deflection. It appears that when a pillar bends under any weight W or $W_1 < W$ the deflection is not owing primarily to diversity of compression, but to a sliding motion of the particles in a horizontal direction which results in a displacement of the compressing force to one side of the pillar axis, hence unequal compression of the transverse area of the pillar.

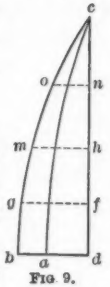

FIG. 9.

In Fig. 9 the curve of deflection $a\,c$ is projected, which is due to unequal compression, and $b\,c$ is the curve of total deflection. This means that $b\,d = \delta_1$ (the actual deflection), and $a\,d$ is that part of the actual deflection which is due to diversity of compression, and it means further that during the process of bending the point d has moved horizontally to b while c remained stationary, also that the points of f, h and n have moved respectively to g, n and o.

If the pillar is assumed to be compressed in a testing machine, c is stationary and d, f, h and n are movable. This motion doubtless begins with any pressure, no matter how small, and when W_1 is very small, $b\,d$ also will be proportionately small. Yet as the line of compression continues to be a straight line between the poles of the pillar axis, and as the pillar axis in the meantime has become a curve, unequal compression is the result. It will be shown hereafter that when the compressing force is very small as compared with the breaking weight, $a\,d$ is also very small as compared with $b\,d$ and when the compressing force becomes the breaking weight, $a\,b = b\,d$.

Resistance to Bending, for $W_1 < W$.

At the breaking point the amount of bending depends upon the bending weight and upon the strength of the material; this means the total potential energy of the material still remaining after a portion of

it has been consumed in resisting vertical stress. The formula $\delta = \frac{C - W}{6\,W}$ means that a pillar bends at the breaking point as much as the resistance of the material will permit and inversely as the bending force. Deflection at the breaking point is an element in determining the length of the pillar, and deflection is as the square of the pillar length limited by the radius, which is also as the square of the pillar length limited by the deflection. Elasticity does not enter into determining the breaking deflection. But when it is intended to compare bending and resistance to bending for weights less than the breaking weight, elasticity becomes an element in comparing the condition of it for a weight $W_1 < W$ with a corresponding condition at the breaking point.

If resistance to bending under the weight $W_1 < W$ be called X then

$$X : C - W = W_1\,E : W\,E_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Now the radius expressed in terms of elasticity (E) is

$$R = \frac{E}{2\,p\,(C - W)}$$

$$\text{Deflection } \delta = \frac{C - W}{6\,W}$$

The pillar length is $L^2 = 8\,R\,\delta$; hence

$$L^2 = \frac{8\,E}{12\,p\,W} = \frac{2\,E}{3\,p\,W} \text{ and } L_1{}^2 = \frac{2\,E_1}{3\,p\,W_1}, \text{ and}$$

$$L^2 : L_1{}^2 = \frac{E}{W} : \frac{E_1}{W_1} = E\,W_1 : E_1\,W \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Comparing equations (6) and (7), $X : C - W = L^2 : L_1{}^2$ where L is the pillar length, and L_1 the length of a pillar of which W_1 is the breaking weight.

Tables Nos. 7 A and 7 B give the X as computed by the above formula for various weights less than the breaking weight, also the respective radii and deflections deduced from the value.

It will be seen that in all cases, whatever the value of W, that

$$\delta\,R = \delta_1\,R_1 = \delta_{11}\,R_{11} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \delta_n = R_n$$

Example.—To find the value of X or pX for a solid 1-in. square cast-iron pillar 9.5 ins. long, of which 48 000 lbs. is the breaking weight, when loaded in the center of resistance with a weight of 24 000 lbs.

The length of a pillar which breaks under a load of 24 000 lbs. is 19.4 diameters $= L_1$, hence $L_1{}^2 = 376.36$, $C - W = 48\,000$ lbs., and $p\,(C - W) = 29\,400$; hence

$$pX : 29\,400 = 90.25 : 376.36 \text{ and } pX = 7\,050 \text{ and } X = 11\,510.$$

To Find the Radius.—The radius R may be found by the formula—

$$R = \frac{2 - [C_p\,(W + p\,(C - W)) + C_p\,(W - p\,(C - W))]}{2\,[C_p\,(W + p\,(C - W)) - C_p\,(W - p\,(C - W))]}$$

by substituting $W_1 = 24\,000$ lbs. for W and $7\,050 = p\,X$ for $p\,(C - W)$.

$$R_1 = \frac{2 - (C_p\,(24\,000 + 7\,050) + C_p\,(24\,000 - 7\,050)}{2\,(C_p\,(24\,000 - 7\,050) - C_p\,(24\,000 - 7\,050)}$$

$$R_1 = \frac{2 - (C_p\,16\,950 + C_p\,31\,050)}{2\,(C_p\,31\,050 - C_p\,16\,950)}$$

$$R_1 = \frac{2 - (0.002897 + 0.001451)}{2\,(0.002897 - 0.001451)} = 690.1.$$

To Find the Deflection.—The breaking radius of a pillar 9.5 ins. in length as previously described is $R = 67.7$, and the breaking deflection $\delta = 0.16666$, hence $\delta\,R = 11.3$ and $\frac{R\delta}{R_1} = \frac{11.3}{690.1} = 0.01637$, the deflection due to the weight $W_1 = 24\,000$ lbs.

The deflection may be computed however from the quantity pX without reference to the radius by the formula $\delta_1 : \delta = C_1 : C$, where C_1 is the difference of compression between that of the intrados and extrados, and C is the compression of $W + p\,(C - W) - (W - p\,(C - W))$, hence

$$\delta_1 = \frac{\delta\,C_p\,(W_1 + pX) + C_p\,(W_1 - pX)}{C_p\,(W + p\,(C - W)) - C_p\,(W - p\,(C - W))},$$

In this case $W + pX = 24\,000 + 7\,050 = 31\,050.$

and $W_1 - pX = 24\,000 - 7\,050 = 16\,950.$

By Table No. 1, compression of $31\,050 = 0.002897$

" " $16\,950 = 0.001450$

and the difference between them $= 0.001447$

By Table No. 1 may also be found

$$C_p\,(W + p\,(C - W)) - C_p\,(W - p\,(C - W)) =$$
$$0.016683 - 0.001606 = 0.014977$$

and as $\delta = \frac{1}{6}$, hence $\delta_1 = \frac{1 \times 0.001447}{6 \times 0.014977} = 0.0161$

The difference (0.00027) between this result and the previous one is owing to the fact that pillar lengths are developed only to the extent of one decimal, which is ordinarily quite sufficient for all practical purposes.

Table No. 7 Å gives the values of $\frac{L^2}{L_1{}^2}$, of X and pX, the radii and

deflections of various weights less than the breaking weight of a solid square cast-iron pillar 9.5 in length.

Table No. 7 A shows the deflection of the

Breaking	weight,	$W = 48\,000$ lbs.	to be	0.16666
"	"	$W_1 = 39\,070$ lbs.	"	0.08333
"	"	$W_1 = 24\,000$ lbs.	"	0.01655
"	"	$W_1 = 12\,000$ lbs.	"	0.00528
"	"	$W_1 = 4\,000$ lbs.	"	0.00155

Table No. 7 B gives the same data for a square solid wrought-iron pillar 19.6 diameters in length.

The deflection under loads less than the breaking load may also be computed directly from the resistance at the breaking point of the pillar $\frac{C - W}{6}$ as follows:

Resistance to bending under lesser weights than the breaking weight, as previously shown, is $X = (C - W)\,\frac{L^2}{L_1^2}$,

where L is the pillar length in question and L_1 is the breaking length of a pillar which breaks under W_1, the weight assumed as $W_1 < W$.

Let w be a force applied transversely upon the pillar at its middle, then $\frac{x}{6} = \frac{w\,L}{4}$ and $w = \frac{2\,X}{3\,L} = \frac{2}{3}\,\frac{(C - W)\,L^2}{L_1^2\,L} = \frac{2}{3}\,\frac{(C - W)\,L}{L_1^2}$ ..... (8)

If w is considered as a transverse load at the middle of the pillar, and the pillar is considered as a beam resting at the top and bottom, its deflection is $\delta_1 = \frac{w\,L^3}{48\,I\,E}$, and if the pillar as heretofore is solid and 1 in. square, I (mom. inertia) $= \frac{1}{12}$ and $\delta_1 = \frac{w\,L^3}{4\,E}$ ..... (9)

Substituting the value of w from equation (8)

$$\delta_1 = \frac{2}{3}\,\frac{(C - W)\,L^4}{4\,k\,L_1^2} = \frac{(C - W)\,L^4}{6\,E\,L_1^2}$$

Example.—Given the solid 1-in. square wrought-iron pillar 19.6 ins. in length, the breaking load of which is $W = 26\,000$ lbs.; its breaking deflection $\delta = 0.16666$; resistance to crushing, $C = 52\,000$ lbs.; hence, $C - W = 26\,000$ lbs.; what is the deflection of this pillar when bending under a load of 12 000 lbs.?

The length of a pillar L_1 that breaks under 12 000 lbs. is 38.8 diameters, and $L_1^2 = 1\,505.44$.

L being 19.6, hence $L^4 = 14757.9$. The average compression of wrought iron within the limit of elasticity is 0.000045 per 1 000 lbs., hence $E = 22\,220\,000$ and $\delta_1 = \frac{(C - W)\,L^4}{6\,E\,L_1^2} = \frac{26\,000 \times 14\,757.9}{6 \times 22\,220\,000 \times 1\,505.44} =$ 0.019. Table No. 7 B gives 0.01828.

Deflections of safe loads may also, within the limit of elasticity, be derived directly from the breaking deflection by the following method:

$$\text{If } \delta_1 = \frac{(C - W)\,L^4}{6\,E\,L_1^2} \text{ then } \delta = \frac{(C - W)\,L^4}{6\,E_1\,L^2} = \frac{(C - W)\,L^2}{6\,E_1} \text{ and}$$

$$\delta_1 : \delta = \frac{(C - W)\,L^4}{6\,E\,L_1^2} : \frac{(C - W)\,L^2}{6\,E_1} = \frac{L^2}{E} : \frac{L_1^2}{E_1}$$

Approximately the modulus of elasticity of the last 1 000 lbs. of W is to the modulus of elasticity of the last 1 000 lbs. of W_1 as $E : E_1$.

Fig. 10 explains the reason for this statement. Let $a\ b\ c\ d$ be the trapezoid which represents the resistance at the middle of the pillar to vertical load and to bending; then $f\ g$ is the bending force and the triangles $a\ m\ g$ and $n\ g\ d$ represent the resistance to bending, and $a\ m$ and $n\ d$ express a number of pounds, and m and n the mean of this number of pounds. Hence the average of the thousands of pounds located between d and a is to be found at n, and the modulus of elasticity of the 1 000 lbs. at n is also approximately the average of the moduli of the various thousands of pounds contained between d and a. This is approximately but not absolutely true, but the error involved is small and on the safe side.

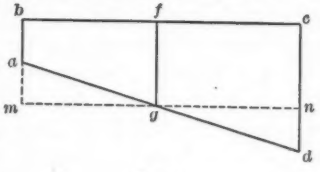

FIG. 10.

Now if the last 1 000 lbs. of $n = (W)\ 1\,000$, and of $m_1 = (W_1)\ 1\,000$, then $E = \frac{(W)\ 1\,000}{C_p\ 1\,000}$ and $E_1 = \frac{W_1\ 1\,000}{C_p\ 1\,000}$ and $\delta_1 : \delta = L^2\,(C_p\,W_1)\,\frac{1\,000}{C_p\,1\,000} : L_1^2\,(C_p\,W)\,\frac{1\,000}{C_p\,1\,000}$ and $\delta_1 = \delta\,\frac{L^2}{L_1^2}\,\frac{C_p\ 1\,000}{C_p\ 1\,000}\,\frac{(W_1)}{(W)}$

Example.—What is the deflection caused by a weight of 12 000 lbs. upon a solid 1-in. square wrought-iron pillar 19.6 ins. long? $W = 26\,000$ lbs. and $C_p\ (W)\ 1\,000 = 0.00007_1$ $L^2 = 374.16$; $L_1^2 = 1\,505.44$; $C_p\,(W)\ 1\,000 = 0.00004$; $\delta = \frac{1}{6}$; hence $\delta_1 = \frac{1}{6}\,\frac{374.16}{1505.44}\,\frac{4}{7} = 0.02367$. Table No. 7 gives a deflection of 0.01828, hence the safe load given in Table No. 9, being 10 720 lbs., would be reduced by 388 lbs.

What is the deflection caused by 16 000 lbs. upon a solid cast-iron pillar 9.5 ins. long and 1 in. square as heretofore described? The breaking weight is 48 000 lbs. and the compression due to the last 1 000 lbs., as given in Table No. 1, column 3, is 0.00025. That of the last 1 000 lbs. of 16 000 lbs. is 0.000083, $L^2 = 90.25$ and $L_1^2 = 696.96$. The breaking deflection $\delta = \frac{1}{6}$ hence

$$\delta_1 = \frac{\delta\ L^2\ C_p\ W_1\ 1\,000}{L_1^2\ C_p\ W\ 1\,000} = \frac{1 \times\ 90.25 \times 0.000083}{6 \times 696.96 \times 0.000250} = 0.00835$$

which exceeds the deflection given in Table No. 7 (0.00800) by 0.00035, hence the safe load is reduced by 33.6 lbs.

An experimental test taken from the reports of the Watertown Arsenal confirms the correctness of deflections δ_1 of weights less than the breaking weight. The report of the Watertown Arsenal of 1888–89 gives on page 733 the deflections of a round hollow cast-iron pillar of 6.35 ins. outer and 4.24 ins. inner diameter under various loads when compressed in the testing machine. It ultimately failed by triple flexure. The length of the pillar is 13.09 or 25.3 diameters, hence the pillar considered as hinged at the ends which corresponds with it is 12.65 diameters long. The radius of gyration of this pillar expressed in terms of the diameter is found to be 0.030236 (in inches, 1.92), hence the length in terms of the radius of gyration is 40.9. Table No. 8 gives the safe load of 14 380 lbs. for this length. The deflection of a pillar 11.8 diameters long (which length in diameters, as given in Table No. 8, corresponds with a length in radius of gyration of 40.9) caused by a load of 14 380 lbs., may be computed by the formula $X : (C - W) = L^2 : L_1^2$ where L is the length of a pillar which has a breaking weight of 14 380 lbs. and which will be found by Table No. 1 to be 29.1; hence $X : 54\,000^* = 139.24 : 846.8$ when $X = 8\,877$ lbs. and $pX = 5\,437$. Hence the radius of the pillar in question 11.8 diameters long bent by a load of 14 380 lbs. is 1 011 ins. and $\frac{L^2}{8\,R} = \delta_1 = \frac{139.24}{8\,088} = 0.0172$.

The metal area of the pillar reported by the Watertown Arsenal is 17.28 sq. ins., hence the total load of the pillar corresponding to the safe load 14 380 lbs. is 248 486 lbs. The report gives deflections at the

* Breaking weight of a pillar 11.8 diameters long is $W = 38\,000$ lbs.
$T = 16\,000$ "
$W + T = 54\,000$ lbs.

middle of the pillar in two directions, horizontal and vertical, for 240 000 lbs., of 0.04 and 0.06 in., respectively, and for 260 000 lbs., of 0.05 and 0.07 in. From this data may be deduced the deflection of 248 436 lbs. as 0.056 and 0.084 in. The true deflection the resultant of the vertical and horizontal deflections is $\delta_1 = \sqrt{0.056^2 + 0.084^2} = 0.101$. Divide this by the diameter of the pillar, 6.35, and it will be found that the deflection $\delta_1 = 0.016$.

Table No. 7 A gives the radii and deflections pertaining to various weights (W_1) from nil to 48 000 lbs., which is the breaking weight of a solid 1-in. square cast-iron pillar 9.5 ins. in length.

Table No. 7 B gives the radii and deflections pertaining to various weights (W) from nil to 26 000 lbs., which is the breaking weight of a solid 1-in. square wrought-iron pillar 19.6 ins. long.

If it is intended that the maximum strain of the pillar shall not exceed 16 000 lbs., one-sixth of the ultimate strength of the material, and that no load bending the pillar can be considered to be a safe load which results in a maximum strain greater than 16 000 lbs., it is clear that a load of 15 232 lbs. would certainly constitute a safe load, inasmuch as the resistance to the bending moment does not exceed 768 lbs. In fact, the deflection caused by a weight of 15 232 lbs. is less than 0.008, hence the bending moment is less than 128 lbs. It will be seen by Table No. 8 that the safe load for the pillar described previously is 15 300 lbs.

Table No. 8 has been computed in the manner indicated. A load W_1 is assumed as probable to constitute a safe load of a given pillar, and its deflection is computed by the method given on page 389; then the correctness of the assumed load is tested by the formula $W_1 + 6\delta_1 W_1 = MS$, wherein MS means the maximum strain at the intrados. If $MS > 16\,000$ lbs., then W_1 must be smaller than the W_1 first assumed, and if $MS < 16\,000$ lbs., then W_1 is greater. Column 1 in Table No. 8 gives the lengths of pillars expressed in diameters, and column 2 in terms of the radius of gyration, for convenient reference in computing safe loads for pillars of all forms. Column 3 gives the safe loads, and column 4 the respective breaking loads.

It will be noticed that for pillars less than seven diameters in length the safe load is one-fourth the breaking load, and less when the pillar is very short. From seven to about twelve diameters in length, the safe load varies from one-fourth to one-third of the breaking load. From

12 to 25 diameters the safe load varies from one-third to one-half the breaking load, and from 25 to 50 diameters the safe load is nearly one-half the breaking load. The significance of the relation of safe loads to breaking loads becomes more apparent by reversing the above statement as follows :

Breaking loads approximate safe loads (which cause a given maximum strain) in magnitude in a ratio related to the length of the pillar.

With a pillar of infinite length the safe load and the breaking weight must both be nil, and with a pillar of inappreciable length, say, a thin plate, the safe load multiplied by the factor of safety is equal to the breaking weight.

Table No. 9 gives for wrought-iron pillars all the data contained in Table No. 8 for cast-iron pillars.

Example.—Find from Table No. 8 the safe load per inch of metal of a round, hollow cast-iron column 20 ft. long and 12 ins. outer and 10 ins. inner diameter.

The radius of gyration is $R_g = \sqrt{\frac{D^2 + d^2}{16}} = \sqrt{\frac{144 + 100}{16}} = \frac{17}{4} =$ 4.25 ins. The length of the pillar in inches is $20 \times 12 = 240$ and length in terms of the radius of gyration is $L_g = 56.5$. Table No. 8 gives the safe load per inch of metal of a pillar 56.5 radii of gyration in length as 12 000 lbs.

It may be noted here that cast-iron pillars not exceeding 28 diameters or 97 radii of gyration in length are not subject to tension at the extrados under a safe load, as given in Table No. 8, when the load is placed in the center of gravity of the pillar. Inequalities and other probable defects of castings are assumed to be covered by the factor of safety.

Breaking loads for cast-iron pillars as given in Table No. 1 and for wrought-iron pillars as given in Table No. 2, also the respective safe loads, as given in Tables Nos. 8 and 9, being computed on the assumption that the load is applied in the center of gravity of the pillar, it is essential that this should be the case accurately, inasmuch as slight deviations cause material differences in their magnitude. A cast-iron pillar 10 ins. in diameter and 11.9 ft. in length ($L = 14.3$) will break under a load of 32 000 lbs. per inch metal area, when the load is placed in the center of gravity of the pillar. When placed 1 in. to one side of the center it will break under 21 150 lbs., and when placed 0.5 in. off the center of gravity of the pillar the breaking load is 26 050 lbs. or 19% less than when exactly in the center.

Brickwork.

The report of tests of the strength of structural material made at the Watertown Arsenal during the year 1884 gives on page 92 the compression of a common hard-brick pier, 8 ins. square and 16.48 ins. high, laid in lime mortar composed of 1 part lime and 3 parts sand, 15 months old when tested.

If brickwork laid up with good cement (say, 1 part of the best Rosendale and 2½ parts of sand) is used in ordinary building, at least six months expire before it is loaded to the full extent of the safe load assumed. The resistance to compression of brickwork so laid may be considered fully as great as that of brickwork laid up in lime mortar 15 months old. Its compressibility will probably be less. It is doubtless safe to assume that it is no greater.

Table No. 10 gives compression in parts of 1 for loads up to 2 000 lbs. per square inch, as deduced from the Watertown experiments, and shows a sufficient uniformity to warrant the acceptance of a modulus of elasticity of 160 000 without appreciable error in practice.

Let 2 000 lbs. be the breaking weight (it is 2 440 lbs. in the Watertown test) and 300 lbs. the maximum stress permissible under a safe load, also let the cohesion of the cement mortar be ignored as a medium of resistance to tension, then one-sixth the diameter becomes the greatest possible deflection at the breaking point. In other words, the breaking point occurs whenever the stress at the intrados amounts to 2 000 lbs. or the stress at the extrados is nil.

With uniform elasticity the radius $R = \frac{E}{2\,p\,(C-W)}$ and as $\delta = \frac{C-W}{6\,W}$ and $L^2 = 8\,\delta\,R$, hence, $L^2 = \frac{2\,E}{3\,p\,W}$ and as $E = 160\,000$, then $L^2 = \frac{174\,000}{W}$ and $W = \frac{174\,000}{L^2}$.

The stress at the intrados under a safe load is 300 lbs. up to a pillar length of 13.2 diameters, where the breaking load is 1 000 lbs. In pillars of greater lengths the stress at the intrados cannot exceed twice the amount of the safe load without causing tension at the extrados.

For lengths in terms of the diameters, L, the square of the respective lengths, L^2, breaking loads per square inch of sectional area, breaking deflections, safe loads per inch area and deflections under them, see Table No. 11.

It will be noticed that for lengths greater than 13.2 diameters the deflection of safe loads is intended to remain constant.

The weights of floors supported in the center of resistance of a wall or pier constitute a part of the superincumbent weight, pound for pound. It frequently happens, however, that floor beams or girders are so placed that they constitute an eccentric load, and if n be the eccentricity, such floor loads should be multiplied by $(6\,n + 1)$ in order to ascertain the value of the centric load which represents them. This rule is approximately correct in view of the minute deflections under safe loads as applied in practical building.

Yellow Pine Pillars.

Table No. 12 gives the safe loads for yellow pine pillars based upon $E = 1\,600\,000$, computed as described previously.

For knots and sap due allowance is to be made, and 25% may be added for pillars for temporary use.

Summary.

For convenience in use, the leading formulas deduced in the preceding discussion are printed here, together with the notation.

Notation.—C is the ultimate resistance of the material to compression.

T is the ultimate resistance of the material to tension.

W " breaking weight.

W_1 is a weight less than the breaking weight.

δ is the breaking deflection.

δ_1 " deflection caused by W_1.

R " radius at the breaking point.

R_1 " " with the load W_1.

L " length of the pillar in terms of the diameter.

d " diameter, taken as 1 in the paper. Hence the moment of inertia I_1 is $\frac{1}{12}$ for a solid square section, and the moment of resistance is $\frac{1}{6}$.

C_p is the amount of compression.

R_g " radius of gyration.

p " coefficient 0.6125.

E_c is the modulus of elasticity for compression.

E_t " " " tension.

n " eccentricity, expressed by a fraction of the diameter.

x and y are the coefficients of stress in eccentric loads.

Formulas.—Breaking weights and safe loads are computed upon the assumption that pillars are compressed in the center of resistance and are subject to single flexure, except in the case of eccentric loading.

1. Deflection at the breaking point.

$$\delta = \frac{C - W}{6\ W}$$

2. Radius at the breaking point corresponding to given deflections.

$$R = \frac{2 - (C_p\ (W + p\ (C - W)) + C_p\ (W - p\ (C - W)))}{2\ (C_p\ (W + p\ (C - W)) - C_p\ (W - p\ (C - W)))}$$

3. Breaking length.

$$L = \sqrt{8\ \delta\ R}$$

With eccentric loading.

4. Compression at the ends of the pillar.

$$x = 6\ n + 1 \qquad y = 6\ n - 1.$$

5. Breaking weight expressed in ultimate resistance of the material and centric breaking loads.

$$W_1 = \pm \sqrt{\left(\frac{C + W + 6\ n\ W}{1}\right)^2 - C\ W} + \left(\frac{C + W + 6\ n\ W}{2}\right)$$

$$W_1 = \pm \sqrt{\left(\frac{W - 6\ n\ W - T}{2}\right)^2 + T\ W} + \left(\frac{W - 6\ n\ W - T}{2}\right)$$

6. Resistance, to bending when W_1 is less than W.

$$X = (C - W)\frac{L^2}{L_1^2}$$

7. Deflection, when W_1 is less than W.

$$\delta_1 = \frac{(C - W)\ L^4}{6\ E\ L_1^2} \text{ or } \delta_1 = \delta\ \frac{L^2\ C_p\ 1\ 000\ (W_1)}{L_1^2\ C_p\ 1\ 000\ (W)}$$

8. Brickwork.

$$E = 160\ 000. \quad W = \frac{2\ E}{3\ p\ L^2} = \frac{174\ 000}{L^2} \quad R = \frac{E}{2\ p\ (C - W)} = \frac{130\ 60\ 0}{C - W}.$$

9. Yellow pine posts.

$$E = 1\ 600\ 000 \qquad W_1 = \frac{1\ 740\ 000}{L^2} \qquad R = \frac{1\ 306\ 000}{C - W}.$$

TABLE No. 1.—Breaking Weights for Round-Ended Solid 1-In. Square Cast-Iron Pillars, Loaded in the Center.

Breaking weight, W.	Compression due to W.	Compression due to the last 1 000 pounds.	Strain at the middle of the column. Compression, —. Tension, +.	Deflection, δ.	Average strain at the intrados, $W + p\,(C - W)$.	Average strain at the extrados, $W - p\,(C - W)$.	Radius, R.	Length, L.
96 000	0.026000	.....	—96 000	.....	96 000	96 000	.....	.....
95 000	0.025333	0.000667	—94 000	0.00175	95 612	94 388	1 259.2	4.2
94 000	0.024736	0.000597	—92 000	0.00345	95 223	92 775	670.2	4.2
93 000	0.024155	0.000581	—90 000	0.00537	94 837	91 163	462.9	4.4
92 000	0.023590	0.000565	—88 000	0.00724	94 449	89 551	354.6	4.4
91 000	0.023038	0.000552	—86 000	0.00916	94 062	87 938	288.8	4.5
90 000	0.022493	0.000545	—84 000	0.01111	93 675	86 325	244.2	4.6
89 000	0.021955	0.000538	—82 000	0.01311	93 287	84 713	213.2	4.7
88 000	0.021424	0.000531	—80 000	0.01515	92 900	83 100	189.1	4.8
87 000	0.020900	0.000524	—78 000	0.01724	92 512	81 488	170.5	4.8
86 000	0.020383	0.000517	—76 000	0.01938	92 125	79 875	155.7	4.9
85 000	0.019873	0.000510	—74 000	0.02157	91 737	78 263	143.7	4.9
84 000	0.019370	0.000503	—72 000	0.02381	91 350	76 650	133.5	5.0
83 000	0.018874	0.000496	—70 000	0.02610	90 962	75 038	125.1	5.1
82 000	0.018385	0.000489	—68 000	0.02846	90 575	73 425	117.8	5.1
81 000	0.017903	0.000482	—66 000	0.03086	90 187	71 813	111.6	5.2
80 000	0.017428	0.000475	—64 000	0.03333	89 800	70 200	106.3	5.3
79 000	0.016960	0.000468	—62 000	0.03797	89 412	68 588	101.5	5.5
78 000	0.016499	0.000461	—60 000	0.04060	89 025	66 975	97.9	5.6
77 000	0.016045	0.000454	—58 000	0.04113	88 637	65 363	93.8	5.6
76 000	0.015598	0.000447	—56 000	0.04386	88 250	63 756	90.6	5.6
75 000	0.015158	0.000440	—54 000	0.04666	87 862	62 138	87.7	5.7
74 000	0.014725	0.000433	—52 000	0.05000	87 475	60 525	85.1	5.8
73 000	0.014299	0.000426	—50 000	0.05251	87 087	58 913	82.3	5.9
72 000	0.013880	0.000419	—48 000	0.05585	86 700	57 300	80.7	6.0
71 000	0.013468	0.000412	—46 000	0.05879	86 312	55 688	78.8	6.2
70 000	0.013063	0.000405	—44 000	0.06190	85 855	54 145	77.5	6.2
69 000	0.012665	0.000398	—42 000	0.06522	85 537	52 463	75.6	6.2
68 000	0.012274	0.000391	—40 000	0.06872	85 150	50 850	74.3	6.2
67 000	0.011890	0.000384	—38 000	0.07211	84 762	49 238	73.1	6.4
66 000	0.011513	0.000377	—36 000	0.07575	84 375	47 625	72.0	6.6
65 000	0.011143	0.000370	—34 000	0.07949	83 987	46 013	71.0	6.7
64 000	0.010718	0.000362	—32 000	0.08333	83 600	44 400	70.2	6.8
63 000	0.010426	0.000355	—30 000	0.08730	83 212	42 788	69.5	6.9
62 000	0.010078	0.000348	—28 000	0.09140	82 825	41 175	68.9	7.0
61 000	0.009737	0.000341	—26 000	0.09590	82 437	39 563	68.4	7.2
60 000	0.009403	0.000334	—24 000	0.10000	82 050	37 950	68.1	7.3
59 000	0.009076	0.000327	—22 000	0.10482	81 662	36 338	67.6	7.5
58 000	0.008756	0.000320	—20 000	0.10920	81 275	34 725	67.5	7.6
57 000	0.008443	0.000313	—18 000	0.11404	80 887	33 113	67.3	7.8
56 000	0.008137	0.000306	—16 000	0.11905	80 500	31 500	67.2	8.0
55 000	0.007838	0.000299	—14 000	0.12406	80 112	29 888	67.2	8.1
54 000	0.007546	0.000293	—12 000	0.13000	79 725	28 275	67.2	8.3
53 000	0.007261	0.000285	—10 000	0.13522	79 337	26 663	67.3	8.5
52 000	0.006983	0.000278	— 8 000	0.14102	78 950	25 050	67.4	8.7
51 000	0.006712	0.000271	— 6 000	0.14706	78 562	23 438	67.4	8.9
50 000	0.006448	0.000264	— 4 000	0.15333	78 175	21 825	67.5	9.1
49 000	0.006191	0.000257	— 2 000	0.15986	77 787	20 213	67.7	9.3
48 000	0.005941	0.000250	0 000	0.16666	77 400	18 600	67.7	9.5
47 000	0.005698	0.000243	+ 2 000	0.17375	77 012	16 988	67.8	9.7
46 000	0.005460	0.000238	+ 4 000	0.18116	76 625	15 375	68.0	9.9
45 000	0.005232	0.000228	+ 6 000	0.18888	76 237	13 763	68.1	10.1
44 000	0.005014	0.000218	+ 8 000	0.19700	75 850	12 150	68.3	10.2
43 000	0.004806	0.000208	+10 000	0.20542	75 462	10 538	68.5	10.6
42 000	0.004608	0.000198	+12 000	0.21429	75 075	8 925	68.6	10.8
41 000	0.004418	0.000190	+14 000	0.22358	74 687	7 313	68.7	11.1
40 000	0.004235	0.000183	+16 000	0.23333	74 300	5 700	68.9	11.3

TABLE No. 1—(*Continued*).

Breaking weight, W.	Compression due to W.	Compression due to the last 1 000 pounds.	Strain at the middle of the column. Compression, —; Tension, +	Deflection, δ.	Average strain at the intrados, $W + p\,(C - W)$.	Average strain at the extrados, $W - p\,(C - W)$.	Radius, R.	Length, L.
39 000	0.004059	0.000176	*—94 000	0.28500	72 687	5 313	72.2	11.6
38 000	0.003890	0.000169	—92 000	0.23684	71 075	4 925	74.3	11.8
37 000	0.003728	0.000162	—90 000	0.23874	69 462	4 538	79.8	12.3
36 000	0.003533	0.000155	—88 000	0.24074	67 850	4 150	83.7	12.7
35 000	0.003425	0.000148	—86 000	0.24285	66 237	3 763	87.9	13.0
34 000	0.003283	0.000142	—84 000	0.24510	64 625	3 375	93.2	13.5
33 000	0.003147	0.000136	—82 000	0.24747	63 012	2 988	97.6	13.9
32 000	0.003016	0.000131	—80 000	0.25000	61 400	2 600	103.3	14.3
31 000	0.002891	0.000125	—78 000	0.25268	59 787	2 213	109.1	14.8
30 000	0.002771	0.000120	—76 000	0.25555	58 175	1 825	115.3	15.3
29 000	0.002657	0.000114	—74 000	0.25862	56 562	1 438	121.5	15.8
28 000	0.002547	0.000110	—72 000	0.26190	54 950	1 050	128.7	16.4
27 000	0.002441	0.000106	—70 000	0.26543	53 337	663	136.4	17.0
26 000	0.002337	0.000104	—68 000	0.26920	51 725	275	144.7	17.6
25 000	0.002235	0.000102	—66 000	0.27333	50 112	112	153.7	18.3
24 000	0.002134	0.000101	—64 000	0.27777	48 500	500	170.4	19.4
23 000	0.002034	0.000100	—62 000	0.28261	46 887	887	173.8	19.8
22 000	0.001935	0.000099	—60 000	0.28800	45 175	1 175	185.8	20.6
21 000	0.001837	0.000098	—53 000	0.29365	43 662	1 662	196.5	21.5
20 000	0.001740	0.000097	—56 000	0.30000	42 050	2 050	208.9	22.4
19 000	0.001644	0.000096	—54 000	0.30700	40 437	2 437	221.4	23.3
18 000	0.001549	0.000095	—52 000	0.31480	38 825	2 825	234.6	24.3
17 000	0.001455	0.000094	—50 000	0.32335	37 212	3 212	248.5	25.3
16 000	0.001362	0.000093	—48 000	0.33333	35 600	3 600	262.3	26.4
15 000	0.001270	0.000092	—46 000	0.34444	33 987	3 987	276.9	27.6
14 000	0.001179	0.000091	—44 000	0.35715	32 375	4 375	291.7	28.8
13 000	0.001089	0.000090	—42 000	0.37154	30 772	4 772	306.6	30.1
12 000	0.001000	0.000089	—40 000	0.38888	29 150	5 150	322.2	31.8
11 000	0.000912	0.000088	—38 000	0.40909	27 537	5 537	337.8	33.2
10 000	0.000825	0.000087	—36 000	0.43333	25 925	5 925	353.9	35 0
9 000	0.000739	0.000086	—34 000	0.46300	24 312	6 312	371.3	37.0
8 000	0.000654	0.000085	—32 000	0.50000	22 700	6 700	389.7	39.4
7 000	0.000570	0.000084	—30 000	0.54760	21 087	7 087	409.9	42.3
6 000	0.000487	0.000083	—28 000	0.61111	19 775	7 475	425.9	45.6
5 000	0.000404	0.000083	—26 000	0.70000	17 832	7 832	460.8	50.0
4 000	0.000322	0.000082	—24 000	0.83333	16 250	8 250	486.0	56.9
3 000	0.000240	0.000082	—22 000	1.05555	14 637	8 637	521.0	66.3
2 000	0.000195	0.000081	—20 000	1.50000	13 025	9 025	535.7	80.0
1 000	0.000079	0.000080	—18 000	2.83330	11 412	9 412	584.4	115.1

* Strains at this point change from the extrados to the intrados, where heretofore the strain has been 96 000. At the extrados the strain continues at +16 000 to the end of the table.

TABLE No. 2.—Breaking Weights for Round-Ended Solid 1-In. Square Wrought-Iron Pillars, Loaded in the Center.

Breaking weight, W.	Compression due to W.	Compression due to the last 1 000 pounds.	Strain at the middle of the column. $-$ Compression, $+$ Tension,	Deflection, δ.	Average strain at the extrados.	Average strain at the intrados.	Radius, R.	Length, L.
52 000	0.00800	.....	−52 000	.....	−52 000	52 000	.....	.....
51 000	0.00749	0.00051	−50 000	0.0033	−50 388	51 612	1 654.2	6.6
50 000	0.00701	0.00048	−48 000	0.0066	−48 776	51 224	863.4	6.7
49 000	0.00655	0.00046	−46 000	0.0102	−47 163	50 837	605.7	7.0
48 000	0.00611	0.00044	−44 000	0.0140	−45 550	50 450	473.2	7.4
47 000	0.00569	0.00042	−42 000	0.0177	−43 938	50 062	394.0	7.4
46 000	0.00529	0.00040	−40 000	0.0217	−42 325	49 675	346.5	7.7
45 000	0.00491	0.00038	−38 000	0.0259	−40 713	49 287	313.8	8.0
44 000	0.00455	0.00036	−36 000	0.0303	−39 100	48 900	290.1	8.4
43 000	0.00421	0.00034	−34 000	0.0349	−37 488	48 512	273.3	8.7
42 000	0.00389	0.00032	−32 000	0.0397	−35 875	48 125	261.4	9.1
41 000	0.00359	0.00030	−30 000	9.0447	−34 263	47 737	254.7	9.5
40 000	0.00331	0.00028	−28 000	0.0500	−32 650	47 350	250.0	10.0
39 000	0.00305	0.00026	−26 000	0.0555	−31 038	46 962	250.0	10.5
38 000	0.00281	0.00024	−24 000	0.0614	−29 425	46 575	250.0	11.1
37 000	0.00259	0.00022	−22 000	0.0676	−27 813	46 187	250.0	11.6
36 000	0.00239	0.00020	−20 000	0.0741	−26 200	45 800	250.0	12.1
35 000	0.00221	0.00018	−18 000	0.0810	−24 588	45 412	253.6	12.8
34 000	0.00205	0.00016	−16 000	8.0884	−22 975	45 025	257.6	13.5
33 000	0.00191	0.00014	−14 000	0.0960	−21 363	44 637	261.7	14.1
32 000	0.00178	0.00013	−12 000	0.1042	−19 750	44 250	265.0	14.8
31 000	0.00166	0.00012	−10 000	0.1129	−18 128	43 872	270.4	15.6
30 000	0.00155	0.00011	− 8 000	0.1222	−16 525	43 475	272.1	16.3
29 000	0.00145	0.00010	− 6 000	0.1322	−14 913	43 087	277.8	17.1
28 000	0.00136	0.00009	− 4 000	0.1426	−13 400	42 600	284.2	18.0
27 000	0.00128	0.00008	− 2 000	0.1543	−11 688	42 312	286.0	18.7
26 000	0.00121	0.00007	0 000	0.1666	−10 075	41 925	288.4	19.6
25 000	0.00115	0.00006	+ 2 000	0.1800	− 8 463	41 537	292.6	20.5
24 000	0.00110	0.00005	+ 4 000	0.1944	− 6 850	41 150	295.4	21.4
23 000	0.00105	0.00005	+ 6 000	0.2101	− 5 238	40 762	301.5	22.5
22 000	0.00100	0.00005	+ 8 000	0.2272	− 3 625	40 375	306.2	23.5
21 000	0.00095	0.00005	+10 000	0.2460	− 2 013	39 987	309.0	24.6
20 000	0.00090	0.00005	+12 000	0.2666	— 400	39 600	310.0	25.7
19 000	0.00085	0.00005	+14 000	0.2900	+ 1 212	39 212	318.0	27.1
18 000	0.00080	0.00005	+16 000	0.3148	+ 2 825	38 825	320.0	28.4
17 000	0.00075	0.00005	+18 000	0.3431	+ 4 437	38 437	323.5	29.7
16 000	0.00070	0.00005	+20 000	0.3750	+ 6 050	38 050	326.3	31.3
15 000	0.00065	0.00005	+22 000	0.4111	+ 7 662	37 662	328.0	32.8
14 000	0.00060	0.00005	+24 000	0.4524	+ 9 275	37 275	329.6	34.5
13 000	0.00055	0.00005	+26 000	0.5000	+10 887	36 887	331.8	36.4
12 000	0.00050	0.00004	+28 000	0.5555	+12 500	36 500	332.0	38.4
11 000	0.00046	0.00004	+30 000	0.6212	+14 112	36 112	329.9	40.5
10 000	0.00041	0.00004	+32 000	0.7000	+15 725	35 725	329.7	42.9
9 000	0.00037	0.00004	+34 000	0.8000	+17 337	35 337	327.6	45.7
8 000	0.00032	0.00004	+36 000	0.9166	+18 950	34 950	326.6	48.9
7 000	0.00028	0.00004	+38 000	1.0714	+20 562	34 562	325.5	52.8
6 000	0.00024	0.00004	+40 000	1.2777	+22 175	34 175	323.5	57.5
5 000	0.00020	0.00004	+42 000	1 5666	+23 778	33 778	322.4	63.5
4 000	0.00016	0.00004	+44 000	2.0000	+25 400	33 400	318.3	71.3
3 000	0.00012	0.00004	+46 000	2.7222	+27 010	33 010	313.4	82.5
2 000	0.00008	0.00004	+48 000	4.1666	+28 625	32 625	304.8	100.7
1 000	0.00004	0.00004	+50 000	8.5000	+30 240	32 240	295.0	141.6
500	0.00002	0.00004	+51 000	17.1666	+32 040	33 040	280.9	196.4

TABLE No. 3.—Breaking Loads of Round-Ended Solid 1-In. Square Cast-Iron Pillars, Loaded in the Center.

Compression, — Tension, +

Breaking weight, W.	Strains at the middle of the pillar at extrados to *, afterward at intrados.	Deflection, δ.	Average strain at the intrados, $\frac{W}{+p(O-W)}$.	Average strain at the extrados, $\frac{W}{-p(O-W)}$.	Radius, R.	Length, L.
96 000	—96 000	.....	.....	.....	.....	.....
95 000	—94 000	0.0020	95 635	94 365	1 230.8	4.0
94 000	—92 000	0.0040	95 273	92 727	643.7	4.4
93 000	—90 000	0.0064	94 909	91 091	455.5	4.8
92 000	—88 000	0.0088	94 545	89 455	349.1	4.9
91 000	—86 000	0.0104	94 182	87 818	268.6	4.8
90 000	—84 000	0.0124	93 675	86 275	243.1	4.8
89 000	—81 630	0.0144	93 287	84 486	.....	.....
88 000	—79 250	0.0168	92 900	82 640	.....	.....
87 000	—76 730	0.0184	92 512	80 710	.....	.....
86 000	—74 300	0.0200	92 125	78 834	.....	.....
85 000	—72 150	0.0220	91 737	77 130	133.5	4.8
84 000	—70 000	0.0240	91 350	75 425	.....	.....
83 000	—67 500	0.0265	90 962	73 510	.....	.....
82 000	—65 000	0.0290	90 575	71 590	.....	.....
81 000	—62 500	0.0305	90 187	69 670	.....	.....
80 000	—60 000	0.0324	89 800	67 750	96.4	5.0
79 000	—57 500	0.0344	89 412	65 220	.....	.....
78 000	—55 000	0.0368	89 075	63 910	.....	.....
77 000	—52 500	0.0400	88 640	62 000	.....	.....
76 000	—50 000	0.0432	88 250	60 075	81.2	5.3
75 000	—48 800	0.0480	87 826	57 730	77.8	5.4
74 000	—43 500	0.0528	87 475	54 700	73.5	5.5
73 000	—39 500	0.0580	87 090	52 480	71.3	5.7
72 000	—35 000	0.0632	86 700	49 338	.....	.....
71 000	—30 400	0.0697	86 312	46 133	65.9	6.0
70 000	—25 300	0.0752	85 925	42 620	.....	.....
69 000	—19 700	0.0826	85 537	38 800	61.2	6.3
68 000	—14 000	0.0900	85 150	34 925	59.8	6.5
67 000	—10 000	0.0975	84 762	32 090	.....	.....
66 000	— 6 000	0.1000	84 375	29 250	.....	.....
65 000	— 2 200	0.1054	83 987	26 535	58.6	7.0
64 000	+ 1 000	0.1104	83 600	24 190	58.1	7.1
63 000	+ 4 500	0.1144	83 170	21 660	58.0	7.2
62 000	+ 8 000	0.1184	82 825	19 125	.....	.....
61 000	+11 500	0.1224	82 437	16 590	57.7	7.5
60 000	+15 000	0.1264	82 050	14 063	57.5	7.6
59 000	+16 000	0.1304	81 540	13 060	.....	.....
	*				.....	.....
58 000	—94 600	0.1344	80 420	12 675	.....	.....
57 000	—93 290	0.1372	79 230	12 290	58.5	8.0
56 000	—91 975	0.1400	78 035	11 900	63.8	8.4
55 000	—90 670	0.1432	76 850	11 510	66.0	8.7
54 000	—89 350	0.1464	75 650	11 125	68.3	8.9
53 000	—88 040	0.1492	74 460	10 740	70.6	9.1
52 000	—86 725	0.1520	73 270	10 350	71.6	9.3
51 000	—85 420	0.1540	72 080	9 960	75.7	9.6
50 000	—84 100	0·1560	70 890	9 575	79.0	9.9
49 000	—82 640	0.1572	69 500	9 190	82.1	.....
48 000	—81 175	0.1584	68 320	8 800	85.1	10.0
47 000	—80 020	0.1608	67 225	8 410	88.0	10.6
46 000	—78 850	0.1632	66 120	8 025	91.1	10.9
45 000	—77 540	0.1652	64 930	7 640	95.3	11.2
44 000	—76 225	0.1672	63 740	7 250	99.3	11.5
43 000	—74 920	0.1676	62 550	6 860	102.4	11.7

TABLE No. 3—(*Continued*).

Breaking weight, W.	Strains at the middle of the pillar at extrados to^{0}, afterward at intrados.	Deflection, δ.	Average strain at the intrados, $\frac{W}{} + p\,(C - W)$.	Average strain at the extrados, $\frac{W}{} - p\,(C - W)$.	Radius, R.	Length, L.
42 000	—73 600	0.1680	61 350	6 475	106.5	11.9
41 000	—72 290	0.1692	60 165	6 090	112.1	12.3
40 000	—70 975	0.1704	58 970	5 700	115.7	12.5
39 000	—69 700	0.1720	57 800	5 310	120.5	12.8
38 000	—68 350	0.1736	56 590	4 925	125.7	13.2
37 000	—66 040	0.1760	54 790	4 540	134.4	13.8
36 000	—65 725	0.1784	54 200	4 150	137.0	14.0
35 000	—64 420	0.1808	53 020	3 760	143.0	14.4
34 000	—63 100	0.1824	51 820	3 375	149.5	14.7
33 000	—61 790	0.1856	50 630	2 990	156.1	15.2
32 000	—60 475	0.1888	49 440	2 600	164.4	15.5
31 000	—59 170	0.1928	48 250	2 210	172.1	16.4
30 000	—57 850	0.1968	47 060	1 825	179.1	16.8
29 000	—56 540	0.2004	45 970	1 440	187.0	17.0
28 000	—55 230	0.2044	44 730	1 050	196.2	17.8
27 000	—53 920	0.2084	43 490	660	205.4	18.5
26 000	—52 600	0.2128	42 290	375	215.1	19.1
25 000	—51 300	0.2192	41 110	112	224.4	19.8
24 000	—50 000	0.2256	39 920	500	234.2	20.5
23 000	—48 500	0.2336	38 620	890	245.2	21.4
22 000	—47 000	0.2416	37 090	1 275	260.0	22.4
21 000	—46 000	0.2504	36 310	1 660	269.0	23.2
20 000	—45 000	0.2592	35 310	2 050	275.0	23.9
19 000	—44 250	0.2708	34 460	2 470	282.3	24.7
18 000	—43 500	0.2824	33 620	2 825	290.5	25.6
17 000	—42 550	0.2980	32 650	3 210	298.2	26.6
16 000	—41 600	0.3136	31 680	3 600	306.9	27.7
15 000	—40 800	0.3316	30 800	3 990	312.9	28.8
14 000	—40 000	0.3496	29 910	4 375	320.4	29.9
13 000	—39 000	0.3724	28 910	4 762	329.4	31.3
12 000	—38 000	0.3953	27 910	5 150	338.4	32.7
11 000	—36 650	0.4320	26 710	5 540	350.2	34.8
10 000	—35 300	0.4688	25 500	5 925	362.5	36.9
9 000	—33 650	0.5128	24 100	6 310	374.4	39.9
8 000	—32 000	0.5568	22 700	6 700	390.6	41.7
7 000	—30 150	0.6160	21 180	7 090	410.1	44.9
6 000	—28 300	0.6752	19 660	7 475	431.3	48.2
5 000	—26 650	0.7504	18 260	7 860	448.7	51.9
4 000	—25 000	0.8256	16 860	8 259	473.5	55.9

TABLE No. 4.—Breaking Weights of Round-Ended Solid 1-In. Square Wrought-Iron Pillars, Loaded in the Center.

Compression, — Tension, +

Breaking weight, W.	Strains at the middle of the pillar.	Deflection, δ.	Average strain at the intrados, $\frac{W}{} + p\,(C - W)$.	Average strain at the extrados, $\frac{W}{} - p\,(C - W)$.	Radius, R.	Length, L.
52 000	52 000					
51 000	49 900	0.0040	51 610	50 330	1 600.8	7.1
50 000	47 800	0.0080	51 225	48 650	827.5	7.2
49 000	45 500	0.0120	50 840	46 860	558.1	7.3
48 000	43 000	0.0160	50 450	44 940	424.7	7.3
47 000	40 600	0.0208	50 030	42 960	351.4	7.6
46 000	38 200	0.0256	49 675	41 220	310.3	7.7
45 000	36 100	0.0304	49 290	39 550	285.2	8.3
44 000	34 000	0.0352	48 900	37 875	268.3	8.6
43 000	31 500	0.0428	48 512	35 960	252.0	9.2
42 000	28 000	0.0504	48 125	33 425	337.7	9.7
41 000	25 000	0.0600	47 740	31 200	230.6	10.5
40 000	20 500	0.0696	47 350	28 160	222.9	11.1
39 000	15 000	0.0788	46 960	24 300	218.8	11.7
38 000	4 000	0.0890	46 575	12 275	199.0	11.8
37 000	12 000	0.0996	46 190	6 999	195.9	12.2
36 000	16 000	0.1112	45 800	4 150	197.7	13.2
35 000	19 600	0.1228	45 410	1 430	199.5	14.0
34 000	22 000	0.1344	45 025	300	202.4	14.7
33 000	24 000	0.1488	44 640	2 910	204.4	14.7
32 000	25 800	0.1632	44 250	3 220	210.0	16.5
31 000	27 000	0.1812	43 860	4 525	213.8	17.6
30 000	28 000	0.1992	43 475	5 525	218.8	18.6
29 000	29 400	0.2164	43 090	6 770	226.3	19.7
28 000	30 000	0.2336	42 700	7 525	227.8	20.6
27 000	31 000	0.2522	42 310	8 525	232.1	21.6
26 000	31 500	0.2708	41 925	9 220	237.7	22.7
25 000	31 750	0.2952	41 540	9 760	248.6	24.0
24 000	32 000	0.3169	41 150	10 300	249.3	25.1
23 000	32 250	0.3440	40 760	10 840	255.3	26.5
22 000	32 500	0.3684	40 375	11 380	261 3	27.7
21 000	33 000	0.3932	39 990	12 075	267.0	28.9
20 000	33 500	0.4180	39 600	12 770	271.3	30.1
19 000	33 750	0.4462	39 210	13 310	279 7	31.6
18 000	34 000	0.4744	38 825	13 850	285.3	32.9
17 000	34 500	0.5299	38 440	14 240	292.2	35.1
16 000	35 000	0.5855	38 050	15 240	297.3	37.3
15 000	35 500	0.6419	37 660	15 930	302.7	39.4
14 000	36 000	0.6983	37 275	16 625	308.3	41.5
13 000	36 500	0.7503	36 890	17 320	311.0	43.2
12 000	37 000	0.8023	36 500	18 010	319.2	45.2
11 000	37 500	0.8934	36 110	18 700	325.4	48.2
10 000	38 000	0.9846	35 725	19 400	330.0	50.9
9 000	38 300	1.0374	35 340	19 970	334.2	52.6
8 000	38 600	1.0902	34 950	20 540	340.0	54.4
7 000	39 300	1.3979	34 560	21 360	343.4	62.0
6 000	40 000	1.7056	34 175	22 175	346.2	68 7
5 000	40 750	2.2910	33 790	23 020	348.9	80.0
4 000	40 500	2.7526	33 400	23 870	351.8	88.0

TABLE No. 5.—Eccentric Breaking Loads for Various Eccentricities (n = a Fraction of the Diameter) of Round-Ended Solid 1-In. Square Cast-Iron Pillars.

$$W_1 = \pm\sqrt{\left(\frac{W - T - 6\,n\,W}{2}\right)^2 + T\,W} + \frac{W - T - 6\,n\,W}{2}$$

Central breaking load, W.	Length, L.	Breaking loads for—				
		$n = 0.1$.	$n = 0.3$.	$n = 0.5$.	$n = 0.7$.	$n = 1.0$.
84 000	5.0	46 500	13 840	7 040	4 640	3 060
72 000	6.0	45 130	13 260	7 000	4 590	3 040
62 000	7.0	41 500	12 670	6 760	4 530	3 010
56 000	8.0	36 680	12 260	6 650	4 480	3 000
50 000	9.1	30 350	11 800	6 530	4 430	2 980
45 000	10.1	27 850	11 360	6 400	4 380	2 950
41 000	11.1	25 930	10 970	6 290	4 330	2 930
37 000	12.3	23 800	10 540	6 150	4 270	2 900
35 000	13.0	22 680	10 310	6 080	4 250	2 880
32 000	14.3	21 150	10 000	6 000	4 200	2 880
30 000	15.3	20 000	9 670	5 860	4 130	2 840
27 000	17.0	18 350	9 220	5 700	4 050	2 800
24 000	19.4	16 600	8 770	5 470	3 960	2 760
21 000	21.5	14 920	8 190	5 300	3 940	2 710
17 000	25.3	12 500	7 360	4 950	3 660	2 620
13 000	30.1	10 000	6 350	4 470	3 410	2 500
10 000	35.0	8 000	5 430	4 000	3 120	2 340
5 000	50.0	4 350	3 000	2 700	2 160	1 860

TABLE No. 6.—Eccentric Breaking Loads for Various Eccentricities (n = a Fraction of the Diameter) of Round-Ended Solid 1-In. Square Wrought-Iron Pillars.

$$W_1 = \pm\sqrt{\left(\frac{C + W + 6\,n\,W}{2}\right)^2 - C\,W} + \frac{C + W + 6\,n\,W}{2}$$

Central breaking load, W.	Length, L.	Breaking loads for—				
		$n = 0.1$.	$n = 0.3$.	$n = 0.5$.	$n = 0.7$.	$n = 1.0$.
49 000	7.0	23 930	17 960	11 850	8 550	6 560
40 000	10.0	22 170	13 860	10 310	8 270	6 320
36 000	12.1	21 180	13 400	10 000	8 100	6 290
30 000	16.3	19 340	12 650	9 672	7 790	6 100
25 000	20.5	17 430	11 800	9 100	7 450	5 980
21 000	24.6	15 600	10 930	8 570	7 090	5 640
18 000	28.4	14 020	10 150	8 070	6 740	5 430
16 000	31.3	12 470	9 480	7 120	6 460	5 240
14 000	34.8	12 090	8 840	6 930	6 100	5 030
12 000	38.3	10 240	8 050	6 690	5 740	4 710
10 000	43.9	8 780	7 140	6 000	5 270	4 420

TABLE No. 7 A.—A Round-Ended Solid 1-In. Square Cast-Iron Pillar. $W = 48\,000$ Lbs., $R = 67.7$ Ins., $\delta = \frac{1}{6}$. When Loaded in the Center with Various Weights $W_1 < W$.

Bending weights, W_1.	$\frac{L^2}{L_1^2}$	X.	pX.	R_1	δ_1	Compression due to bending moment.	$\frac{X}{6}$
48 000	1.00000	48 000	29 400	67.7	0.16666	0.16666	8 000
44 000	.86745	41 637	25 500	88.2	0.12812	0.10000	5 637
40 000	.70679	33 926	20 780	124.9	0.09047	0.05384	3 620
39 070	.6742	32 360	19 820	135.4	0.08333	0.04065	3 256
36 000	.55961	26 860	16 352	185.2	0.06100	0.02700	2 200
32 000	.44134	21 800	12 975	278.4	0.04059	0.01300	1 300
28 000	.33555	16 106	9 855	442.8	0.02552	0.00600	714
24 000	.23979	11 510	7 050	690.0	0.01655	0.00460	397
20 000	.17986	8 613	5 296	977.7	0.01155	0.00177	231
16 000	.12949	6 215	3 807	1 418.5	0.00800	0.00090	128
12 000	.08914	4 280	2 620	2 157.6	0.00528	0.00046	63
8 000	.05913	2 790	1 710	3 458.0	0.00330	0.00015	26
4 000	.02787	1 338	820	7 300.0	0.00155	0.00004	6.2
1 000	.00681	327	200	31 232.0	0.00036	0.00000	0.36

At the breaking point the deflection is 0.16666; hence the bending moment is 48 000 × 0.1666 = 8 000. To bend the pillar to half the breaking deflection, the bending weight is 39 070 lbs., and the bending moment 3 256 lbs. An endwise compression of 24 000 lbs. (one-half the breaking weight) results in a deflection of 0.01655, about $\frac{1}{12}$ of the breaking deflection and the bending moment is 24 000 × 0.01655 = 397, or about the twentieth part of the bending moment at the breaking point.

When the load is 16 000 lbs. the deflection is reduced to 0.008, and the bending moment is 16 000 × 0.008 = 128 lbs., about the sixtieth part of the bending moment at the point of breaking. It may be observed here that the resistance to bending in this case being six times the bending moment, 6 × 128 = 768 lbs., hence the strain at the intrados is 16 768 lbs., consisting of the strain caused by resistance to the bending moment and that of vertical compression caused by the bending weight.

TABLE No. 7 B.—A Solid Round-Ended 1-In. Square Wrought-Iron Pillar, 19.6 Ins. Long, Loaded in the Center with Various Weights $W_1 < W$.

W_1	L.	L^2	$\frac{L_2}{L_1^2}$	pX.	X.	δ_1
26 000	19.6	374.16	.....	15 926	26 000	0.16666
24 000	21.4	457.96	0.81701	13 010	21 240	0.10300
22 000	23.5	552.25	0.67752	10 790	17 600	0.07224
20 000	25.8	665.64	0.56210	8 950	14 610	0.04758
18 000	28.4	806.56	0.46389	7 390	12 060	0 03568
16 000	31.3	979.69	0.38202	6 080	9 932	0.02900
14 000	34.8	1 211.04	0.30895	4 920	8 030	0.02292
12 000	38.8	1 505.44	0.24854	3 960	6 460	0.01828
10 000	43.9	1 927.21	0.19414	3 090	5 040	0.01305
8 000	50.0	2 500.00	0.14096	2 240	3 670	0.00900
6 000	59.6	3 552.16	0.10533	1 670	2 744	0.00406
4 000	74.4	5 535.36	0.06760	1 120	1 760	0.00112
2 000	100.7	10 140.00	0.03689	587	960	0.00091

TABLE No. 7a.—If a Solid 1-In. Square Cast-Iron Pillar is Loaded in Center with Weights Varying from 48 000 Lbs. (its Breaking Weight) to 24 000 Lbs., and if by Any of These Loads it were Bent to One-Half its Breaking Deflection on the Assumption that $\frac{S - W_1}{6\ W_1} = \delta_1$, then $S - W_1$ will Give a Radius Greater than $R_1 = 135.4$. Hence $X > S - W_1$, and there is a Horizontal Shifting of the Particles Composing the Pillar.

W_1	S.	$S - W_1$	$p\ (S - W_1)$	R.
48 000	72 000	24 000	14 600	140.3
44 000	66 000	22 000	13 475	172.7
40 000	60 000	20 000	12 250	219.5
36 000	54 000	18 000	11 025	286.8
32 000	48 000	16 000	9 800	381.1
28 000	42 000	14 000	8 575	514.9
24 000	36 000	12 000	7 300	664.8

TABLE No. 8.—Safe Loads of Round-Ended Solid 1-In. Square Cast-Iron Pillars Loaded in the Center. Maximum Strain = 2 670 Lbs. Tension, 16 000 Lbs. Compression. Given for all Pillar Lengths, both in Diameters and Radii of Gyration.

Lengths in diameters.	Lengths in radii of gyration.	Safe load.	Breaking weights.	Lengths in diameters.	Lengths in radii of gyration.	Safe load.	Breaking load.
6.0	20.7	15 820	.....	28.0	96.9	7 650	.....
6.2	21.5	15 750	72 000	28.8	99.8	7 450	14 000
6.7	23.2	15 660	65 000	29.0	100.4	7 400	.....
7.0	24.2	15 630	.....	30.0	102.9	7 100	.....
7.3	25.3	15 540	60 000	30.1	104.3	7 070	13 000
8.0	27.7	15 510	.....	31.0	107.3	6 800	....
8.3	28.7	15 480	54 000	31.8	110.1	6 640	12 000
8.9	30.8	15 410	51 000	32.0	110.8	6 600	.....
9.0	31.1	15 400	.....	33.0	114.3	6 400	.....
9.5	32.9	15 300	48 000	33.2	115.0	6 360	11 000
9.7	33.5	15 260	47 000	34.0	117.7	6 209	.....
10.0	34.6	15 200	.....	35.0	121.2	5 920	10 000
10.1	35.0	15 150	45 000	36.0	124.7	5 640	.....
10.6	36.7	14 900	43 000	37.0	128.2	5 360	9 000
11.0	38.1	14 700	.....	38.0	131.6	5 080	.....
11.1	38.4	14 660	41 000	39.0	135.0	4 780	.....
11.3	39.1	14 580	40 000	39.4	136.4	4 500	8 000
11.6	40.2	14 460	39 000	40.0	138.5	4 280	.....
11.8	40.9	14 380	38 000	41.0	142.0	4 060	.....
12.0	41.5	14 300	.....	42.0	145.4	3 780	....
12.7	44.0	13 950	36 000	42.3	146.5	3 600	7 000
13.0	45.0	13 800	.....	43.0	148.9	3 450	.....
13.5	46.7	13 510	34 000	44.0	152.4	3 300	.....
14.0	48.4	13 220	....	45.0	155.8	3 150	.....
14.3	49.5	13 080	31 000	45.6	158.0	3 000	6 000
15.0	51.9	12 750	.....	46.0	159.3	2 800	.....
15.3	53.0	12 590	30 000	47.0	162.8	2 600	.....
15.8	54.7	12 580	29 000	48.0	166.2	2 400	.....
16.0	55.3	12 200	.....	49.0	169.7	2 200	.....
16.4	56.8	12 030	28 000	50.0	173.2	2 000	5 000
17.0	58.9	11 780	27 000				
17.6	61.0	11 440	26 000				
18.0	62.3	11 220	.....				
18.3	63.4	11 090	25 000				
19.0	65.8	10 800	.....				
19.4	67.2	10 620	24 000				
19.8	68.6	10 440	23 000				
20.0	69.2	10 350	.....				
20.6	71.3	10 080	22 000				
21.0	72.7	9 900	....				
21.5	74.5	9 700	21 000				
22.0	76 2	9 500	.....				
22.4	77.6	9 380	20 000				
23.0	79.6	9 190	.....				
23.3	80.7	8 920	19 000				
24.0	83.1	8 800	.....				
24.3	84.2	8 710	18 000				
25.0	86.6	8 500	.....				
25.3	87.6	8 410	17 000				
26.0	90.0	8 200	....				
26.4	91.4	8 080	16 000				
27.0	93.5	7 900	.....				
27.6	95.6	7 750	15 000				

To find the safe load (L_{1s}) for eccentric loads.

Let centric safe loads be L_s and eccentric safe loads L_{1s}.

Let centric breaking weights be W and eccentric breaking weights be W_1, then L_{1s}: $L_s = W_1 : W$ and $L_{1s} = \frac{L_s W_1}{W}$

TABLE No. 9.—Safe Loads of Round-Ended Solid 1-In. Square Wrought-Iron Pillars, Loaded in the Center. Maximum Strain = 12 000 Lbs. for all Pillar Lengths Given in Diameters and in Radii of Gyration. Formula for Eccentric Safe Loads in Table No. 8.

Length in diameters.	Lengths in radii of gyration.	Safe load.	Breaking weight.	Length in diameters.	Lengths in radii of gyration.	Safe loads.	Breaking weights.
10.0	34.6	11 810	.....	41.3	143.1	4 970	11 000
10.5	36.4	11 800	39 000	42.0	145.4	4 850	.....
11.0	38.1	11 800	.....	43.0	148.8	4 650	.....
11.6	40.2	11 800	37 000	43.9	152.0	4 470	10 000
12.0	41.5	11 790	.....	44.0	152.4	4 450	.....
13.0	45.0	11 750	.....	45.0	155.8	4 280	.....
13.5	46.8	11 700	34 000	46.0	159.3	4 110	.....
14.0	48.4	11 700	.....	47.0	162.8	3 950	9 000
15.0	51.9	11 600	.....	48.0	166.2	3 880	.....
15.6	54.0	11 550	31 000	49.0	169.7	3 650	.....
16.0	55.4	11 520	.....	50.0	173.2	3 500	.....
16.3	56.5	11 500	30 000	51.0	176.6	3 380	.....
17.0	58.8	11 400	.....	52.0	180.1	3 240	.....
17.1	59.2	11 390	29 000	53.0	183.5	3 100	.....
18.0	62.3	11 200	28 000	54.0	187.0	3 000	.....
18.7	64.8	11 110	27 000	54.5	188.8	2 950	7 000
19.0	65.8	10 900	.....	55.0	190.3	2 900	.....
19.6	67.7	10 720	26 000	56.0	193.8	2 770	.....
20.0	69.2	10 600	.....	57.0	197.4	2 650	.....
20.5	71.0	10 400	25 000	58.0	200.9	2 550	.....
21.0	72.7	10 200	.....	59.0	204.3	2 470	.....
21.4	74.1	10 060	.....	59.6	206.3	2 430	6 000
22.0	76.2	9 850	.....	60.0	207.8	2 400	.....
22.5	78.0	9 700	23 000	61.0	211.3	2 300	.....
23.0	79.6	9 550	.....	62.0	214.7	2 200	.....
23.5	81.7	9 250	22 000	63.0	218.2	2 120	.....
24.0	84.1	9 150	.....	64.0	221.7	2 070	.....
24.6	85.2	8 990	21 000	65.0	224.9	2 020	.....
25.0	87.6	8 880	.....	65.7	227.6	1 990	5 000
26.0	90.0	8 600	20 000	66.0	228.6	1 980	.....
27.0	93.5	8 350	.....	67.0	232.1	1 900	.....
27.1	93.7	8 320	19 000	68.0	235.6	1 850	.....
28.0	96.9	8 070	.....	69.0	239.0	1 800	.....
28.4	98.4	7 960	18 000	70.0	242.5	1 770	.....
29.0	100.4	7 800	.....	71.0	245.9	1 730	.....
29.7	102.7	7 600	17 000	72.0	249.4	1 700	.....
30.0	103.9	7 520	.....	73.0	252.9	1 650	.....
31.0	107.3	7 300	.....	74.0	256.3	1 600	.....
31.3	108.4	7 200	16 000	75.0	259.8	1 550	.....
32.0	110.8	7 000	.....	76.0	263.3	1 500	.....
33.0	114.3	6 780	15 000	77.0	266.7	1 450	.....
34.0	117.7	6 550	.....	78.0	270.2	1 400	.....
34.8	120.5	6 340	14 000	79.0	273.7	1 350	.....
35.0	121.2	6 290	.....	80.0	277.1	1 300	.....
36.0	124.7	6 090	.....	81.0	280.5	1 250	.....
36.7	127.1	5 900	13 000	82.0	284.0	1 200	.....
37.0	128.1	5 820	.....	83.0	284.0	1 250	.....
38.0	131.6	5 650	.....	84.0	290.9	1 100	.....
38.8	134.4	5 470	12 000	85.0	294.4	1 050	.....
39.0	135.0	5 420	.....	86.0	297.9	1 000	.....
40.0	138.5	5 210	.....	87.0	301.3	950	.....
41.0	142.0	5 020	.....	88.0	304.8	900	.....
.....	.....	.....	.....	89.0	308.2	850	.....
.....	.....	.....	.....	90.0	311.7	800	.....

TABLE No. 10.—Compression of Common Hard Brickwork in Accordance with Experiments at the Watertown Arsenal. Modulus of Elasticity = 160,000 Nearly.

Weights. Pounds per square inch.	Compression. In parts of unity.	Weights. Pounds per square inch.	Compression. In parts of unity.
50	.....	1 100	0.00680
100	0.00122	1 150	0.00722
150	0.00164	1 200	0.00764
200	0.00200	1 250	0.00783
250	0.00233	1 300	0.00813
300	0.00267	1 350	0.00855
350	0.00300	1 400	0.00892
400	0.00327	1 450	0.00900
450	0.00352	1 500	0.00928
500	0.00379	1 550	0.00946
550	0.00410	1 600	0.00960
600	0.00437	1 650	0.01000
650	0.00457	1 700	0.01013
700	0.00479	1 750	0.01031
750	0.00505	1 800	0.01056
800	0.00531	1 850	0.01108
850	0.00566	1 900	0.01116
900	0.00590	1 950	0.01147
950	0.00619	2 000	0.01171
1 000	0.00631	2 050	0.01204
1 050	0.00660	2 100	0.01238

TABLE No. 11.—Piers or Walls of Common Hard Brick laid in best Rosendale Cement with 2 Parts of Sand, or in Portland Cement with 3 Parts of Sand.—Maximum Strain for Safe Loads, 300 Lbs.

Breaking weight.	Length.	L^2	δ	Safe load.	δ_1
2 000	0.0	0.00	0.00000	300	0.00000
1 900	9.5	90.25	0.00868	297	0.00056
1 800	9.8	96.04	0.01847	295	0.00278
1 700	10.1	102.01	0.03000	290	0.00500
1 600	10.4	108.16	0.04160	285	0.00700
1 500	10.8	116.64	0.05600	280	0.01040
1 400	11.1	123.21	0.07124	275	0.01405
1 300	11.6	134.56	0.09000	270	0.01870
1 200	12.0	144.00	0.11080	260	0.02510
1 100	12.3	151.29	0.16000	250	0.03150
1 000	13.2	174.24	0.16666	240	0.04119
900	14.0	196.00	0.16666	225	0.04119
800	15.0	225.00	0.16666	200	0.04119
700	16.0	256.00	0.16666	175	0.04119
600	17.0	289.00	0.16666	150	0.04119
500	18.7	349.69	0.16666	125	0.04119
400	21.0	441.00	0.16666	100	0.04119
300	24.4	595.36	0.16666	75	0.04119
200	29.6	876.16	0.16666	50	0.04119
100	42.0	1 764.00	0.16666	25	0.04119

For sandstone safe loads may be increased by 0 — 40 per cent.
" granite " " " 60 — 160 "
" marble " " " 40 — 100 "
" limestone " " " 10 — 25 "

TABLE No. 12.—YELLOW PINE, SQUARE WOODEN POSTS, STRAIGHT GRAINED, PERFECTLY DRY AND LOADED IN THE CENTER.—WHEN NOT DRY, ONE-HALF THE SAFE LOAD TO BE TAKEN.—MAXIMUM STRAIN FOR SAFE LOADS, 800 LBS.

Length in diameters.	Breaking weight.	Deflection.	Safe load.	Deflection.	Safe load.	Safe load.
0	8 000	.....	800	.....	.....	.....
15	7 800	.00428	795	.0010	.....	.....
16	6 875	.02727	790	.0040	.....	.....
17	6 075	.05281	770	.0075	.....	.....
18	5 425	.07910	755	.0125	.....	.....
19	4 900	.10544	720	.0175	.....	.....
20	4 400	.13636	695	.0225	.....	.....
21	3 975	.16834	675	.0300	.....	.....
22	3 650	.20000	650	.0375	.....	.....
23	3 325	.21158	625	.0450	.....	.....
24	3 050	.27050	605	.0550	540	.....
25	2 825	.30531	580	.0650	460	.....
26	2 625	.34127	555	.0750	400	.....
27	2 400	.39000	530	.0850	320	.....
28	2 250	.42600	505	.0975	250	.....
29	2 075	.47590	475	.1100	220	460
30	1 950	.51710	460	.1225	190	375
35	1 400	.78571	350	.1975	90	150
40	1 100	1.0454	290	.2900	50	90
45	875	1.2810	240	.3900	30	60
50	700	1.7380	200	.4975	20	40
55	575	2.1521	165	.6150	10	30
60	500	2.5	135	.7300	5	20

DISCUSSION.

Mr. Thomson. GEORGE H. THOMSON, M. Am. Soc. C. E.—The author states early in the paper that "it is reasonable to assume that at the intrados of the pillar at the point of greatest deflection a strain has been set up which is as great as the stress causing rupture in a specimen of the material, the height or vertical length of which specimen does not materially exceed its smallest diameter." While it may be reasonable to assume this, the validity of the assumption is questionable. The speaker did not care to deny it, because of the logical fact that from the denial of an antecedent nothing follows, but he believed that its negation was conceivable.

Mr. Berg. WALTER G. BERG, M. Am. Soc. C. E.—Any attempt to obtain the true column formula should be welcomed by engineers, architects and designers, whether working in iron, steel, timber or other structural materials, and the author deserves, therefore, the thanks of the profession for his efforts to contribute to the subject. The deduction of a true column formula is one of the most complex problems in applied mechanics. The search for it is an attractive and alluring study, but it will probably prove as futile, in the sense of arriving at an absolutely final, theoretical solution of the question as the attempt to solve the perpetual-motion problem. When the fact is generally recognized that the subject is beyond the bounds of strictly theoretical investigations, then engineers will have advanced a great step in the matter. They will abandon deductive reasoning, and confine themselves to inductive methods. All their energies will be concentrated on physical tests as the basis of empirical formulas, which will become practically scientifically accurate as the number and value of reliable test results increase. This method is in perfect harmony with the general trend of column-formula investigations during the last century. Starting with the theoretical expressions of Euler and Tredgold, there has been a constant endeavor to extend and improve them by establishing coefficients to accord with additional test results as fast as obtained.

The author presents apparently theoretical formulas, but which, in fact, are premised on certain assumptions as to the physical behavior of the material, and subsequently supplemented by coefficients obtained from actual tests. The result is, therefore, a nominally theoretical solution dependent upon physical assumptions and results from practical experience. The value of the formulas presented can be best determined by comparing them with test results, especially of full-size pieces. It appears from Figs. 3 and 4 that the author's formulas for cast and wrought iron compare favorably with Gordon's formulas, and the results of a limited number of small-size tests made by Hodgkinson. This coincidence is not conclusive, be-

Mr. Berg. cause Gordon's formulas were based chiefly on Hodgkinson's work. It would be very interesting to see how the author's formulas would compare with the valuable test results of iron columns, especially of full-size specimens, obtained within recent years.

The ultimate breaking values for yellow pine columns, given by the author in Table No. 12, do not compare favorably with the plotted results of over 100 full-size tests of yellow pine columns and of empirical formulas established from these tests. It may be that the agreement in the case of iron would be better on account of the greater homogeneity of the material, but for yellow pine the author's formula fails to agree with actual facts, as clearly demonstrated by Fig. 11. This diagram shows that, as compared with test results and recognized empirical formulas, the author's values for yellow pine are too low for very long columns (which may be a good feature), and entirely too high for very short columns, while they are about correct for columns between 18 and 22 diameters.

The author's formula is $W_1 = 1\,740\,000 \div L^2$, where L is the length of the post in terms of the diameter and W_1 is a "weight less than the breaking weight." The values given in the second column of Table No. 12 under the heading "breaking weight" are, however, based apparently on the formula quoted. The curve in the diagram representing these values is hyperbolic in form. The author drops the formula arbitrarily at 15 diameters; if the curve is continued below 15 diameters, the unreliability of the formula from a practical point of view is shown by the fact that the breaking values for very short posts would be in excess of the crushing strength of short blocks of yellow pine. In other words the author's general form of equation

$$\text{Breaking weight} = \text{a constant} \div \text{square of length}$$

is not in accord with the results of tests, and differs radically from all other formulas known to the speaker, which give when plotted either a straight line or a curve of conchoidal shape.

In the following formulas, which are plotted on the diagram, W is the breaking weight in pounds per square inch.

The formulas of William H. Burr, M. Am. Soc. C. E., are based on tests at the Watertown Arsenal of full-size sticks of the average quality in practical use. They are:

$$W = 5\,800 - 70\,\frac{l}{d} \quad \text{for} \quad \frac{l}{d} > 20.$$

$$W = 4\,400 \qquad \text{for} \quad \frac{l}{d} < 20.$$

The formula proposed by James H. Stanwood, Assoc. M. Am. Soc. C. E., is based on the Watertown tests and on tests of full-size sticks made by Prof. G. Lanza. This formula is:

$$W = 4\,250 - 43\tfrac{1}{2}\,\frac{l}{d}$$

Mr. Berg.

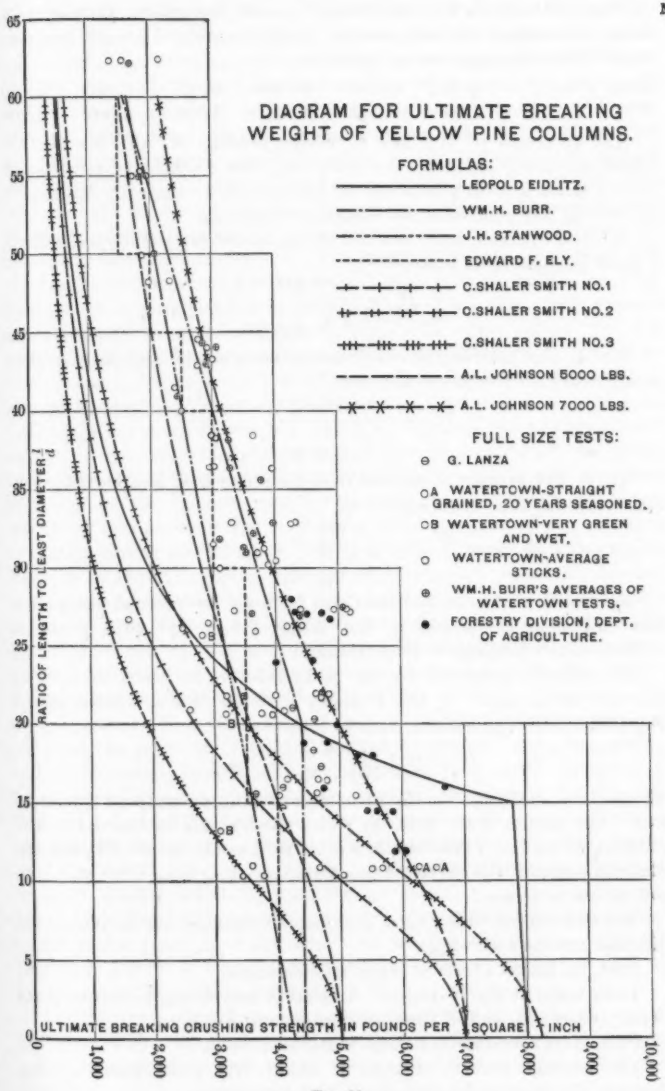

FIG. 11.

Mr. Berg. Prof. Edward F. Ely's rule based on the Watertown tests gives a series of averages between various limiting ratios of length to diameter. These averages are as follows;

Ratio	0–15	15–30	30–40	40–45	45–50	50–60
W.....	4 000	3 500	3 000	2 500	2 000	1 500

The formulas of the late C. Shaler Smith, M. Am. Soc. C. E., which are really Gordon's formula with the coefficients established from a series of over 1 200 tests of full-size sticks made by Mr. Smith in 1861–62, are plotted on the diagram as follows:

No. 1. For green, half-seasoned sticks, answering to the specification "good merchantable lumber."

$$W = \frac{5\,400}{1 + \frac{l^2}{250\,d^2}}$$

No. 2. For selected sticks, reasonably straight and air-seasoned under cover for two years and over.

$$W = \frac{8\,200}{1 + \frac{l^2}{300\,d^2}}$$

No. 3. For average sticks cut from lumber which had been in open-air service for four years and over.

$$W = \frac{5\,000}{1 + \frac{l^2}{250\,d^2}}$$

Mr. Smith recommended the third formula for general use, and it was employed in preparing the tables for yellow pine posts in "Trautwine's Engineers' Pocketbook."

The formula proposed by Mr. A. L. Johnson is based on tests of full-size sticks made by the Forestry Division of the United States Department of Agriculture; it is as follows:

$$W = F\left(\frac{700 + 15\,C}{700 + 15\,C + C^2}\right)$$

where $C = l \div d$ and F is the ultimate crushing strength of the material. The results of the tests on which the formula is based led Mr. Johnson to assume 7 000 lbs. per square inch as the value of F, and the diagram contains the curves of this formula, using both 7 000 and 5 000 lbs. as the unit base.

The tests of full-size yellow pine posts with flat ends plotted in the diagram are the following:

Prof. G. Lanza's tests of seven mill columns.

Tests made at the Watertown Arsenal of two straight-grained posts twenty years old, and of three very green and wet posts.

Prof. Burr's averages of these Watertown tests.

The average results of tests of about fifty posts made for the Forestry Division of the Department of Agriculture.

HENRY S. PRICHARD, M. Am. Soc. C. E.—The author takes the ground that "safe loads of pillars are not proportional to the breaking weights, but must be referred to a permissible maximum strain." To this view, which he holds in common with a number of recent authors, the speaker had no objection when it is confined to pillars so short that there is no danger of failure under any load which can be regarded as a reasonable possibility, but precisely because he agrees with the author that for columns in which bending occurs, the ratio between extreme fiber strains of different intensities is not the same as the ratio between the loads which produce them, he protested earnestly against making the intensity of the strain produced, instead of the relation of load to the breaking weight, the test of safety for long pillars. It is manifestly a mistake to speak of a load as safe when a small increase, something less than 25%, will cause failure, yet if a certain limiting intensity of strain, constant for all ratios of length to radius of gyration, is made the sole requirement, such loads will be allowed in many of the cases which occur in practice. Mr. Prichard.

The author gives Tables Nos. 1 and 2 "derived from curves based upon the results of Hodgkinson's experiments," in which are given breaking weights from 1 000 to 96 000 lbs. for cast iron and from 500 to 52 000 lbs. for wrought iron, for pillars 1 in. square, and, under the heading "Compression due to W," also gives the distortions in terms of the lengths of the columns, corresponding to such weights, which show, as might be expected, that the distortions are relatively greater for large breaking weights than for small ones. From this he concludes that the distortions due to stresses of the same intensity as the breaking weights are the same as those he has tabulated for the breaking weights, and that therefore the distortions are relatively greater for large stresses than for small ones.

This conclusion is not a logical deduction from the tables cited. A pillar at the moment of failure has the stress distributed unequally, and the distortion therefore is caused by a varying stress whose mean intensity is the same as that of the direct compression, that is, the total load divided by the area. As in most cases some portion of the pillar is strained in excess of the elastic limit, and as the distortion for strain which is in excess of the elastic limit is known to be much greater relatively than for strain which is not, the natural inference is that distortion due to a varying stress whose mean intensity is the quotient of the total load divided by the area is greater than that due to a uniform stress whose intensity is the quotient of the total load divided by the area.

The Pencoyd experiments and those made at Watertown show that for wrought iron the modulus of elasticity is nearly constant through a range of stress from zero to the elastic limit. Among the Pencoyd experiments were some, made for the express purpose of determining

Mr. Prichard.

the elasticity of wrought iron under compression, in which the specimens were secured laterally so that they could not bend. The results of one of these tests were given in a paper by James Christie, M. Am. Soc. C. E., entitled "Experiments on the Strength of Wrought Iron Struts."*

The author does not follow the same method in determining the breaking weight and the safe load, and the methods he does follow are not consistent with each other. The force of this objection may be understood by considering two metals having exactly the same elasticity, one of which would crush at a strain of 52 000 lbs. per square inch, and the other at a strain of 12 000 lbs. per square inch. The load which would cause failure in the pillar made of the weaker metal would cause a strain of 12 000 lbs. on the extreme fiber of the stronger metal, and would therefore be for the stronger metal what the author calls the safe load, and the way to obtain it would be to substitute S for C, and W_1 for W in his formula for breaking loads, assuming its accuracy for the sake of the argument, and adopting his notation.

Example.—Find the length of a 1-in. square pillar when the maximum extreme fiber strain is 12 000 lbs. per square inch and the load is 10 000.

$$\delta_1 = \frac{S - W_1}{6\ W_1} = \frac{12\,000 - 10\,000}{6 + 10\,000} = \frac{1''}{30} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$R_1 = \frac{2 - (C_p\,(W_1 + p\,(S - W_1)) + C_p\,(W_1 - p\,(S - W_1)))}{2\,(C_p\,(W_1 + p\,(S - W_1)) - C_p\,(W_1 - p\,(S - W_1)))} \quad \ldots\ldots (2)$$

$$R_1 = \frac{2 - (0.000469 + 0.000351)}{2\,(0.000469 - 0.000351)} = 8471$$

$$L_1 = \sqrt{8\,\delta_1\,R_1} = \sqrt{8 \times \tfrac{1}{30} \times 8471} = 47.5 \text{ ins.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The breaking weight for this length by Table No. 2 is between 8 000 and 9 000 lbs., which is less than the safe load. This result points inevitably to an error, which the speaker believed to lie in the author's system and not in the reasoning by which the result is deduced from it.

Although this method of obtaining the safe load is a logical deduction from the premises given by the author, and although equation (1) can be derived from $S = W_1\,(1 + 6\delta)$ given on page 372, equation (2) is given on page 386, and the same reasoning which develops $L^2 = 8\,R\,\delta$ on page 382, will develop equation (3), it is not the method he uses and the results are not what he obtains.

At one step in the train of reasoning, the author introduces the proportional equation $S - W_1 : C - W = W_1\,E : W\,E_1$, in which $S - W_1$ is written in place of X. If it is correct for any values of E

* See *Transactions*, Vol. xiii, p. 121.

and E_1 which different materials may have, it is correct for $E = E_1$, in which case it becomes: Mr. Prichard.

$S - W_1 : C - W = W_1 : W$, or $SW - WW_1 = CW_1 - WW_1$, or $SW = CW_1$. Hence the proportion, $S : C = W_1 : W$. This proportion is at variance with the author's remarks,* as follows: "and it appears that the relation of S to C can be as the respective weights W_1 to W only on the assumption that $1 + 6\delta_1 = 1 + 6\delta$, or that $\delta_1 = \delta$. It seems clear therefore that safe loads of pillars are not proportional to their breaking weights."

This discrepancy results from the use of different methods, not in accord, for determining the breaking weight and the safe load as defined by the author.

On the supposition of a varying modulus of elasticity, the formulas which the author develops from what he terms the breaking loads centrally applied, would have a different significance from what he assumes. To interpret them correctly the speaker considered first a pillar under the following conditions: Constant cross-section, homogeneous material, constant modulus of elasticity, and an axis coincident before deflection with the line of thrust.

Let M = the bending moment.
L = the length.
E = the modulus of elasticity.
I = the moment of inertia.
δ = the deflection.
W = the load which will just maintain the pillar in equilibrium when the deflection at the center is δ.
F = a factor which is constant for given conditions of loading and end connections.

The general equation for deflection is $\delta = \frac{M L^2}{F E I}$ ..... (4)

$$M = W\delta \text{ ..... (5)}$$

$$\delta = \frac{W \delta L^2}{F E I} \text{ ..... (6)}$$

$$W\delta = \delta \frac{F E I}{L^2} \text{ or } W = \frac{F E I}{L^2} \text{ ..... (7)}$$

This is Euler's equation, and as he has shown that for a column with hinged ends $F = \pi^2$, it can be written

$$W = \frac{\pi^2 E I}{L^2} \text{ ..... (8)}$$

Let W_1 be any load less than W,
W_2 " " greater than W.
It is apparent from equations (6) and (7):

* See page 372.

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First.—The load W will maintain the column in equilibrium under any deflection.

Second.—If W_1 is substituted for W the deflection will become zero.

Third.—If W_2 is substituted for W the deflection will increase until the elastic limit is reached, which is the beginning of failure.

If E should grow less as the strain increased, in the ratio assumed by the author, it would not affect the accuracy of the deduction that any load $W_1 < \frac{\pi^2 E I}{L^2}$ would produce no deflection, but it would affect the deduction that W would maintain the pillar in equilibrium for any deflection, as the bending moment would no longer be simply $W \delta$, but would be increased by the shifting of the axis caused by the different values of E on the concave and convex sides of the pillar.

Under these conditions a smaller load than that required to start deflection will maintain the pillar in equilibrium for a given deflection, if such deflection has already been produced, and it is for these loads under the supposition of a varying modulus of elasticity that the author's formulas deduce, approximately, the deflections and lengths.

The difference between the actual conditions met with in practice and the ideal ones assumed as the basis of formulas (6), (7) and (8) has a marked influence on the behavior of pillars, and this fact should be considered in proportioning them.

CORRESPONDENCE.

Mr. Morison.

GEORGE S. MORISON, Past-President Am. Soc. C. E.—The paper is valuable for its suggestions rather than for its conclusions. The assumption that failure always occurs because some portion of the material in a pillar is strained beyond its ultimate strength is not correct. Neither is the assumption that the extreme local strain is reached by a combination of direct compressive strains and bending strains always correct. The destruction of compression members is occasioned by a great variety of causes, in which the distortion of material when strained beyond the elastic limit usually plays a greater part than the ultimate strength of the metal. A pillar will be crippled when some portion of its material has been strained beyond the elastic limit, although nothing is broken; but a subsequent pressure considerably less than that which crippled the pillar may increase the crippling so much that actual destruction of metal will occur. Pillars also fail through the buckling of individual parts, and this is the usual source of failure when very thin sections are used.

The class of pillars to which the author has applied his analysis are of so simple section that some of these disturbances are eliminated;

still it is not correct to assume that the pillars bend in a single curve from end to end; the curve is more likely to have two points of contrary flexure and the result would be that the strain which produces rupture and which is determined by the radius of curvature, would occur with a very much less lateral deflection than the author's analysis assumes. Mr. Morison.

The formulas which are the basis of the analysis are empirical formulas based on the result of actual tests. The author takes these as accurate measures of the strength of pillars of dimensions corresponding to the assumed conditions, and calculates how much these pillars must deflect laterally to produce a strain in extreme fibers equal to that required to break the material, this being a tensile or a compressive strain according to the length of pillar. He then decides that the factor of safety should not be applied on the load placed upon the pillar, but on the strain in extreme fibers and calculates what pressure on the ends of the pillar would correspond to the working strain which his factor of safety permits on extreme fibers of metal. His demonstration shows correctly that the longer the pillar, the larger the relative amount of strain due to lateral deflection or bending, and, as his factor of safety is applied upon a resultant of which this strain forms a part, it follows that the factor corresponding to the division of the breaking strength of the whole pillar by the safe load on the whole pillar, which is the usual factor of safety, will be less than the factor corresponding to the breaking strain in extreme fibers divided by the working strain in extreme fibers, which is the author's factor of safety. The result is that as the length of pillar increases, the general factor of safety decreases, the fiber factor remaining constant.

While this may be theoretically correct, it can hardly be considered good practice. It overlooks the fact that a factor of safety is intended to cover a great many uncertainties and irregularities, and the longer a pillar becomes, the greater the opportunities for these irregularities. When a pillar passes a certain length it is likely to break by tension on the convex side, and even safe working loads may produce tension on one side and compression on the other. Furthermore, comparatively small disturbances may reverse the deflection, and the pillar which, under one loading, deflects to the right, may under another, which is equally likely to occur, deflect to the left; the result is that, although the actual strains may never exceed what the author has selected as safe working strains on extreme fibers, they may approach this amount, both as positive and negative strains, and the variation between extremes may be double the maximum in either direction. As it is generally conceded that the measure of variation in strains is at least as important as the measure of maximum strains, it follows that what would be a perfectly safe working strain for a short pillar under compression would be absolutely dangerous in pillars subject to such

Mr. Morison. reversals. Again, the longer the pillar, the greater the liability to distortion from accidental blows; a blow which would be entirely harmless on a pillar 10 ft. long would exert nine times as much effect on a pillar 30 ft. long; and if a load came on before the effect of this blow was removed, disaster might follow.

The paper is perhaps an illustration of the dangers of applying theoretical analysis to empirical formulas based on tests which the analyzer has not himself observed. It might be instructive to make tests of pillars of the precise dimensions which have been considered, and observe how far the action of the pillars under strain corresponds with the changes of shape which have been assumed as the basis of his analysis.

The paper, however, brings out one very important feature, which seems in danger of being neglected in architectural construction, though engineers have long recognized its absolute importance. It is briefly the fact that a pillar is not simply a compression member, but a beam as well, and that no pillar is properly designed in which its duties as a beam are not considered. The fundamental requirement of a beam is that it must be able to resist flexure, and this resistance causes bending strains in the parts on opposite sides of the neutral axis. In order to transmit these strains from the point of application to the flanges and from one flange to the other, it is absolutely necessary that the two sides should be connected by something in the nature of a continuous web. In the case of a solid beam this is simple enough. In the case of a built beam, like a plate girder, it requires care in the design; the web must be continuous, and the rivets connecting the web with the flanges must always be enough to transmit to the flanges all the strains which those flanges may have to carry. If an open web is used it must be of some continuous pattern, so that at every point of application no strain but a longitudinal one is carried by the flanges the shearing strain being taken entirely by the web members. In other words, no pillar is properly designed unless it has a web which forms a continuous bracing like the web of a girder or truss.

If cast-iron pillars are used, and the general opinion of engineers would certainly be against using them at all except for short sections, the best form is undoubtedly the round hollow column, provided it be cast vertically; but whatever form is used, it should be of continuous section.

The forms of wrought-iron pillars which are generally approved may be divided into box and lattice columns. The box column is undoubtedly the strongest, and, except for the difficulty of inspection, the best; whether the Phœnix form be used or the square column, made of two plates and two channels, the result is a column which meets all theoretical requirements and gives excellent results. Latticed columns are all right, provided the lacing is properly proportioned and is continuous; they are now the favorite column in

bridge work, partly for convenience of connections, which are generally in parallel planes, and partly because of accessibility for cleaning and painting; on the other hand, they are less convenient, and comparatively little used in architectural work. Mr. Morison.

Another class of pillars has been made of an H form which meets the necessary requirements. These may be made entirely of channels, entirely of I beams, of a combination of channels and I beams, and of late they have been made very largely of a single web plate and four Z irons. All of these pillars have continuous webs and are well adapted to resist bending; care must be taken that the flanges are thick enough to resist buckling, and the tendency to buckle is very greatly reduced by supporting the flanges at moderate intervals.

Varieties of pillars have been introduced from time to time, but generally discarded before very long, which are without continuous webs, being composed of several longitudinal members connected at intervals only. For short pillars under moderate strains, and especially in those places where there is no tendency to any transverse strain, this form of pillar may be admissible, but even then it cannot be economical, as the transverse unit is a dimension of one of the sections of which the pillar is composed and not the width of the pillar itself. To show the weakness of such a pillar it is only necessary to analyze it in the way in which the writer analyzed his cast-iron pillars; to calculate what bending strain must exist in the flanges of a properly designed pillar of a similar cross-section when the pillar fails and then calculate the web required, both in dimensions and connections, to produce these bending strains in the flanges. A defective pillar of this class, which, unfortunately, is now being quite extensively used, may be cited; it is formed of eight angles, arranged as shown in Fig. 12, and connected at intervals by short bent plates. This pillar is very convenient for connections, and, if the connecting plates were continuous, would be a good pillar. As constructed there is no provision for transmission of bending strains except by the transverse stiffness of the angles, and the pillar must necessarily fail under comparatively small bending strains. It is everywhere an inferior pillar, and if used where its transverse stiffness is relied upon to resist wind pressure, it may be classed as dangerous.

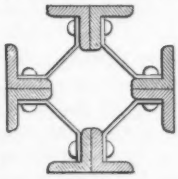

FIG. 12

Passing to another feature, it will be observed that Table No. 8, safe loads of cast-iron pillars, is based on maximum strains of 16 000 lbs. to the inch, while Table No. 9, safe loads of wrought-iron pillars, is based on maximum strains of 12 000 lbs. to the inch. Wrought iron is practically no longer used; but in columns of such length that flexure plays any considerable part in the working strains it certainly is not right to subject cast iron to any greater strains than are permissible on wrought iron.

Mr. Eidlitz. LEOPOLD EIDLITZ, Esq.—It is probable that strain at the middle of the intrados of a pillar at the point of breaking by flexure is greater than the stress causing rupture in a specimen of material when tested to ascertain its resistance to compression, for the reason that the ultimate strain in the pillar is produced in a mathematical point, which is equivalent to the compression of an infinitely thin plate, where all bending strains are eliminated. It is not probable that the ultimate strain of a bent pillar is less than the tested resistance to compression of the material.

For the purpose of analysis, it is therefore "reasonable to assume that at the intrados of a pillar at the point of greatest deflection a strain has been set up which is as great as the stress causing rupture in a cube of the material in question." The author would add that probable errors resulting from this assumption tend to place the results on the safe side.

Beams of a given length and of uniform elasticity when bent under transverse loads deflect proportionately to the load. It is immaterial, therefore, whether in order to ascertain the value of safe loads an assumed factor of safety is applied to the breaking weight of the material, or to its ultimate extreme fiber strain. This is not the case in the bending of a pillar. Given fractions of breaking weight result in variable maximum fiber strains, hence in strength of resistance disproportionate to loads. This condition is enhanced by the fact that matter is not uniformly elastic.

An accurate working formula which includes the results of bending under eccentric loads as differing in its deflection from that caused by centric loads seems desirable, because values under small eccentricities are in practice underated, hence frequently entirely neglected as being covered by the factor of safety; therefore it is stated on page 371.

"The questions examined in this paper are mainly what is the maximum strain in pillars compressed endwise in the center of resistance by loads less than the breaking weights, or by loads not applied in the center of resistance."

In order to ascertain the maximum strains of weights less than the breaking weight, it is necessary to know the resistance at the point of breaking, and inasmuch as the resistance to the breaking weight is the total potential energy of the pillar, after it is compressed vertically by the breaking weight, the breaking weight must first be ascertained in order to know that resistance.

There are two methods pursued to ascertain the breaking weights of pillars of various lengths, neither of which, however, contemplates the use of a coefficient, as is erroneously suggested in the discussion. The method proposed to ascertain breaking weights of various pillar lengths in accordance with Tables Nos. 3 and 4 has been as follows:

For Cast Iron.—Ten diagrams of resistance at the center of the pillar were plotted by the method pursued in Fig. 1. They varied from

compression of 72 000 lbs. at the extrados and 96 000 lbs. at the intrados to nil at the extrados and 96 000 lbs. at the intrados; thence from 10 000 lbs. tension at the extrados and 96 000 lbs. compresssion at the intrados to 16 000 lbs. tension at the extrados and 24 000 lbs. compression at the intrados. Mr. Eidlitz.

For Wrought Iron.—Ten diagrams of resistance at the center of the pillar were plotted, varying from 42 400 lbs. compression at the extrados, and 52 000 lbs. compression at the intrados, to 42 000 lbs. tension at the extrados and 52 000 lbs. compression at the intrados.

From these diagrams the deflection, breaking weights, radii and lengths were computed, and Tables Nos. 3 and 4 formed from curves connecting the various figures of deflection, breaking weight, radius and length.

Tables Nos. 1 and 2 were formed by computing from each breaking weight the deflection, radius and pillar length by the process and formulas stated in the paper. In relation to this matter, attention is drawn to the following paragraph in the paper:

"Under conditions of uniform elasticity, this value of δ is absolutely correct; but inasmuch as elasticity is variable, it would be true only if the horizontal sectional planes drawn transversely to the pillar length were to be curved during the process of bending," which of course is not the case.

When the results of Tables Nos. 1, 2, 3 and 4 were plotted in Figs. 3 and 4, it was found that the curves of Tables No. 1 and 2 agreed as nearly as can be expected with Hodgkinson's experiments (variation of the material being allowed for). Tables Nos. 3 and 4 showed breaking weights somewhat larger for the same pillar lengths, and although those results are doubtless perfectly correct, Tables Nos. 1 and 2 were accepted as the basis of further analysis and calculation, by reason of their being in accord with actual experiments, and for the further reason that the breaking weights are on the safe side.

An examination of Figs. 3 and 4 will show plainly that neither of these curves agree with the curve made in accordance with Gordon's formula. Gordon's curve becomes materially inaccurate in pillar lengths less than 28 to 30 diameters. This is fully in accord with Prof. Rankine's statement* that Gordon's formula gives approximate results only for long rods or pillars.

In the interest of a preference for empirical formulas, which in the language of the discussion "will become practically scientifically accurate as the number and value of reliable test values increase," and in the interest further of separating imperfections of material (in its nature and also in form and magnitude as used in structural parts), also imperfections of methods of testing, from the fundamental cause of bending, which is purely resistance to compression and tension, elasticity in fact, Gordon's and other kindred formulas may be cor-

* See Rankine's "Civil Engineering," p. 237.

Mr. Eidlitz. rected as follows: Gordon's formula gives approximate values for breaking weights of long pillars (of 30 diameters and upwards). It is not available for pillars less than 30 diameters in length. The main cause of this imperfection is to be found in the fact that his coefficient, a, as given by Rankine, is not flexible. It is here intended to show how by the definitions of the deflection δ, the radius R, and the pillar length L, which form the basis of Tables Nos. 1 and 2, the coefficient a may be made flexible; the formula then will cover all pillar lengths with sufficient accuracy for practical use. The error will be less than 1½ per cent.

Gordon's and other kindred empirical formulas are all based upon the equation $\delta\ W = \frac{d^3}{6}(C - W)$ (where δ is the deflection at the breaking point, W the breaking load, and C the ultimate resistance to crushing of the material) rendered in the form $W = \frac{C}{1 + 6\,\delta}$ when the diameter $d = 1$. This value of W is not true, however, in the case of cast-iron pillars of lengths greater than 12 diameters. For these the formula is $W = \frac{T}{6\delta - 1}$ (where T is the ultimate resistance of the material to tension). This fact seems to have been overlooked.

The next effort in composing a working formula has been to express the value of $6\,\delta$, by some $a\,L$*, or $a\,L^2$, where a is a fixed coefficient of L or L^2, and it was assumed that $a\,L$ or $a\,L^2$ may be made to approximate the value of $6\,\delta$ with sufficient accuracy to express the value of W, for long pillars at least.

Now, $\delta = \frac{L^2}{8\,R}$; hence $a = \frac{6}{8\,R}$. The radii of cast-iron pillars between 8 and 50 diameters in length vary between 67 and 460 diameters; consequently in a pillar 8 diameters long with single flexure, $a = \frac{6}{8 \times 67}$ and in a pillar 50 diameters long $a = \frac{6}{8 \times 460}$ (the radii of wrought-iron pillars for 10 to 50 pillar lengths vary between 250 and 327). This shows that to be accurate a must be variable.

It may be observed, however, that R becomes greater or less at some ratio which contains a function of the pillar length, and it may be possible, therefore, to express $\frac{6}{8\,R}$ by $\frac{1}{\pm n\,L \pm c}$ where c and n are constants for each material. Hence, a formula—

$$W = \frac{C}{1 + \frac{L^2}{\pm n\,L \pm c}} \quad \text{or} \quad W = \frac{T}{\frac{L^2}{\pm n\,L \pm c} - 1}$$

may be made sufficiently accurate (within certain given lengths of pillars, at least) of a given material. Upon this principle the following

* L or L^2 may be considered as expressed in diameters or in radii of gyration, then a varies accordingly.

formulas for breaking loads of cast and wrought iron have been computed: Mr. Eidlitz.

For breaking weights, W, for cast-iron pillars from 5 to 11 diameters in length $W = \dfrac{C}{1 + \dfrac{L^2}{245 - 14\,L}}$

where $C = 96\,000$ lbs.

For breaking weights, W, for cast-iron pillars 11 to 30 diameters long $W = \dfrac{T}{\dfrac{L^2}{15.7\,L - 86} - 1}$

where $T = 16\,000$ lbs.

For breaking weights, W, for cast-iron pillars 30 to 50 diameters long $W = \dfrac{T}{\dfrac{L^2}{10\,L + 87} - 1}$

For wrought-iron pillars from 8 to 50 diameters long,

$$W = \frac{C}{1 + \dfrac{L^2}{310 + 2.5\,L}}$$

For compound flexure a must be divided by 4, inasmuch as L^2 becomes $\left(\frac{L}{2}\right)^2$

From the text of the paper it appears clear that outside of the seven formulas enumerated in the summary*, all factors of safety or statements of moduli of elasticity are not an outcome of its deductive demonstrations, but merely assumptions, currently accepted, for the convenience of illustration.

It is well known that yellow pine is not uniformly elastic, and it needs, therefore, no diagram nor extended enumeration of breaking weights and the formulas by which they are computed from the results of various experiments to show that they differ from those resulting from an assumed uniform elasticity which does not exist, but which has in this case been assumed for the sake of brevity in computing safe loads.

When in a structural part the maximum strain is directly proportionate to the weight, it is immaterial whether a determined factor of safety is applied to either; but when, as in the case of pillars, this is not the case, the factor of safety must be applied to the determination of the permissible maximum strain. If it is probable, or even possible, that the computed actual loads may be exceeded by 25% or any other quantity whatever, that quantity must be added to the primarily computed load, or if in the mind of the engineer a doubt still exists as to the safety of an unusually long pillar (and it must be remembered

* See page 397.

Mr. Eidlitz. here that in construction as now practiced, pillars are invariably short, not long, viz., exceeding 20 diameters), or if he is influenced by habit or by a laudable desire to limit deflection to a minute imperceptible quantity, he may in such a special case increase the factor of safety.

Referring again to the quotation on page 423, and applying it to the problematical case of a pillar of the elasticity of wrought iron and an ultimate strength of 12 000 lbs., the method of computing the pillar length must be in accordance with a condition of uniform elasticity as follows:

$$E = \frac{W}{C_p} = \frac{12\,000}{0.0005} = 24\,000\,000$$

$$R = \frac{E}{2p(C-W)}, \quad \delta = \frac{C-W}{6W}$$

$$L^2 = \frac{8E}{12pW} = \frac{1.08843 \times 24\,000\,000}{W} = \frac{26\,122\,000}{W},$$

W being 10 000 lbs. Hence, $L^2 = 2\,612.2$ and $L = 51.1$. According to Table No. 4, which is accurate, the length of the problematical pillar which breaks under 10 000 lbs. is 50.9.

Pages 384 to 387 of the paper, containing a section on "the deflection and radius caused by $W_1 < W$ when placed in the center of resistance," and also Table No. 7a prove conclusively, as stated in the heading of Table No. 7a, that X is greater than $S - W_1$, which means distinctly that $X = S + n - W_1$, when n is a function of W_1.

The discussion contains the sentence: "At one step in the train of reasoning the author introduces the proportional equation $S - W : C - W = W_1 E : W E_1$ in which $S - W_1$ is written in place of X" (evidently by the speaker).

Inasmuch as the equation alluded to in the text of the paper contains the term X and not the term $S - W_1$, this proves the speaker's deductions following the substitution of $S - W_1$ for X to be erroneous.

The formula for the resistance to breaking when W_1 is less than W, $X = (C - W)\frac{L^2}{L_1^2}$, is proved beyond controversy by the fact that when the radius is computed from the value of X, and when the deflection is computed by either of the formulas No. 7*, then $8 R \delta = L^2$, while for any other value of X this is not the case.

Hodgkinson, speaking of Euler's theory of the strength of pillars, says:

"It appeared from the researches of this great analyst, that a pillar of any given dimensions and description of material required a certain weight to bend it, even in the slightest degree; and with less than this weight it would not be bent at all (Acad. de Berlin, 1757). Lagrange, in an elaborate essay in the same work, arrives at the same conclusion. The theory, as deduced from this conclusion, is very

* See page 397.

beautiful, and Poisson's exposition of it in his 'Mecanique,' 2d Edition, Vol. 1, will repay the labor of a perusal. Mr. Eidlitz.

"I have many times sought, experimentally, with great care for the weight producing incipient flexure, according to the theory of Euler, but have hitherto been unsuccessful. So far as I can see, flexure commences with weights far below those with which pillars are usually loaded in practice. It seems to be produced by weights much smaller than are sufficient to render it capable of being measured. I am therefore doubtful whether such a fixed point will ever be obtained, if indeed it exist. With respect to the conclusions of some writers, that flexure does not take place with less than about half the breaking weight; this, I conceive, could only mean large and palpable flexure; and it is not improbable that the writers were in some degree deceived from their having generally used specimens thicker, compared with their length, than have been usually employed in the present effort."

The author would add that he cannot conceive how Euler's formula throws any light upon the maximum strain in pillars compressed endwise in the center of resistance by weights less than the breaking weight, or by weights not applied in the center of resistance.

The discussion contains a considerable amount of matter which has no direct bearing upon the subject of maximum strain under various loads and differing conditions of loading of pillars, which on that account requires no reply on the author's part, excepting inasmuch as it indirectly affects principles involved.

Mr. Morison says: "The assumption that failure always occurs because some portion of the material in the pillar is strained beyond its ultimate strength is not correct." How can matter fail unless strained beyond its strength?

The same writer says also: "It is not correct to assume that pillars bend in a single curve from end to end." They always bend in this way when rounded at the ends as stated in the heading of the tables.

He says also: "The formulas which are the basis of the analysis are empirical formulas based on the results of actual tests." There are no formulas, empirical or otherwise, at the basis of the analysis. The analysis defines deflection and radius by geometrical and mathematical deduction. In another place the same author says: "The paper is perhaps an illustration of the dangers of applying theoretical analysis to empirical formulas based on tests which the analyser has not himself observed." Analysis tends to correct empirical coefficients in the interest of accuracy, a process which cannot be conceived to involve danger to science. The idea that analysis necessarily involves personal observation of tests quoted is not held by the author.

Another criticism by the same writer is:

"Passing to another feature, it will be observed that Table No. 8, safe loads of cast-iron pillars, is based on maximum strains of 16 000 lbs. to the inch, while Table No. 9, safe loads of wrought-iron pillars,

Mr. Eidlitz. is based on maximum strains of 12 000 lbs. to the inch. Wrought iron is practically no longer used; but in columns of such length that flexure plays any considerable part in the working strains it certainly is not right to subject cast iron to any greater strains than are permissible on wrought iron."

Table No. 8 contemplates a permissible maximum tensile strain for cast iron of 2 670 lbs. only.

Engineering handbooks published by various rolling mills and others having the authority of competent engineers compute pillars planed top and bottom or pillars of continuous length held in position by girders and beams abutting upon them and fastened to them with bolts or rivets at every story as subject to compound flexure under a safe load.

Deflections of pillars usually employed in building are exceedingly small under safe loads; for instance deflections of wrought-iron pillars from 12 to 20 diameters long (under loads varying from 11 790 to 10 600 lbs.) are 0.003 to 0.022 diameter.

The movement of the pillar head from a horizontal position to one sufficiently inclined to correspond with the stated deflections is so small that the strains generated are inoperative, because the movement is abundantly practicable within the limits of inaccuracy of construction such as exists in practical building.

If bending were continued to the breaking point, then, no doubt, compound flexure would ensue, but in the absence of loads greater than safe loads, pillars bend with single flexure.

It is also a well-known fact that eccentric loading is under-rated in the absence of a working formula which by one process gives eccentric breaking loads as compared with centric breaking weights and the strength of material.

These considerations have resulted in the analysis of the strength of pillars, and go to show that safe loads are governed by maximum strain, and not by breaking weights, or else many buildings constructed under the old system would show more serious defects than have been discovered as yet, and also that with a table of safe loads at the command of the engineer or architect, no eccentric load, no matter how small, should be neglected on the plea that the factor of safety being applied to weights instead of strains covers a multitude of defects not critically examined.